

(preliminary)

BOTT-TYPE AND EQUIVARIANT SEIBERG-WITTEN FLOER HOMOLOGY I

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ABSTRACT. We construct Bott-type and stable equivariant Seiberg-Witten Floer homology and cohomology for rational homology spheres and prove their diffeomorphism invariance.

TABLE OF CONTENTS

Part I

1. Introduction
2. Preliminaries
3. Seiberg-Witten moduli space over Y
4. Seiberg-Witten trajectories: transversality
5. Index and orientation
6. The temporal model and compactification
7. Bott-type homology
8. Invariance I
- Appendix A Seiberg-Witten Floer homology
- Appendix B. Equivariant Seiberg-Witten Floer homology
- Appendix C. Two analysis lemmas

From Part II

9. Stable Bott-type and stable equivariant homology
10. Invariance II

1. INTRODUCTION

At the very beginning of the development of the new Seiberg-Witten gauge theory it was clear that, at least formally, the celebrated instanton homology theory of A. Floer for 3-manifolds (homology spheres) [8] could be adapted to the Seiberg-Witten set-up. Indeed, the original 4-dimensional Seiberg-Witten equation leads naturally to a 3-dimensional Seiberg-Witten equation via a limit process, as first observed by Kronheimer and Mrowka [12]. To establish a Seiberg-Witten Floer homology for a 3-manifold Y , the obvious idea is to replace flat connections in Floer's set-up by solutions of the 3-dimensional Seiberg-Witten equation on Y (henceforth called *Seiberg-Witten points*), and instanton trajectories by Seiberg-Witten trajectories, which are solutions of the 4-dimensional Seiberg-Witten equation on the infinite cylinder $Y \times \mathbb{R}$. Note that the Seiberg-Witten points are precisely the critical points of the Seiberg-Witten type Chern-Simons functional, and that the Seiberg-Witten trajectories are precisely the trajectories (negative gradient flow lines) of this functional. Hence the said idea amounts to establishing a Morse-Floer theory for the Seiberg-Witten type Chern-Simons functional. However, one encounters

various difficulties when trying to implement this idea. One most serious problem is that Seiberg-Witten Floer homologies for a homology sphere (or rational homology sphere) may depend on the underlying Riemannian metric (cf. e.g. [7]), and hence are generally not diffeomorphism invariants. The purpose of this paper is to resolve this problem.

The trouble, e.g. in the situation of homology spheres, is caused by the trivial Seiberg-Witten point: the trivial connection coupled with the zero spinor field. It is reducible, i.e. it is a fixed point of the action of the group S^1 of constant gauges. Under reasonable perturbations of the Seiberg-Witten equation, this reducible point always survives. To deal with it, one can use suitable perturbations to make it a transversal point for the Seiberg-Witten equation. Then one can construct a Seiberg-Witten Floer homology, see Appendix A. However, one encounters a serious obstruction when trying to compare the homologies for two different perturbation parameters (e.g. metrics). A canonical way of such comparison is to construct chain maps in terms of parameter-dependent Seiberg-Witten trajectories which connect the 3-dimensional Seiberg-Witten equation of one parameter to that of another. We shall call them *transition trajectories*. The said obstruction is the presence of reducible transition trajectories with negative spectral flow of the linearized Seiberg-Witten operator. Such trajectories are not in transversal position and may appear in the compactification of the moduli spaces of transition trajectories between irreducible Seiberg-Witten points. Consequently, the compactified moduli spaces of transition trajectories may be very pathological and cannot be used to define the desired chain maps.

The appearance of such trajectories roughly goes as follows. The spectral flow along a reducible Seiberg-Witten trajectory for a fixed parameter is 1. That along a reducible transition trajectory from a given generic parameter to a nearby one is also 1. When passing from one generic parameter to another through certain degenerate parameters, the spectral flow jumps and becomes negative. Here, typically, Seiberg-Witten Floer homology also jumps.

Bott-type Construction

Since the “ordinary” Seiberg-Witten Floer homologies may not be diffeomorphism invariants for rational homology spheres, we seek an alternative construction. It turns out that the Bott-type set-up above the S^1 quotient is a right one. In other words, we work on the level of quotient by the group of based gauges rather than the full group of gauges. *But even in this set-up, one has to choose the right approach in order to obtain an invariant theory.*

Now, in this alternative set-up, the irreducible part of the moduli space of gauge classes of Seiberg-Witten points consists of finitely many circles, while its reducible part consists of a single point, provided that we choose a generic parameter. These circles and the reducible point are precisely the critical submanifolds of the (Seiberg-Witten type) Chern-Simons functional. Our goal here amounts to establishing a Bott-type Morse-Floer theory for the Chern-Simons functional. The basic strategy is to use the moduli spaces of trajectories between critical submanifolds to send (co)homological chains from one critical submanifold to others, which defines the desired boundary operator for the (co)chain complex. This is a natural extension of Floer’s construction and was first used by Austin-Braam [3] and Fukaya [9] in Floer’s set-up. The former authors use differential forms as chains and cochains, while the latter uses “geometric chains”. We shall adopt the classical singular

chains and cochains. We emphasize that *it is not clear whether the differential form approach leads to diffeomorphism invariants*. On the other hand, an advantage of the singular chain set-up is that it admits integer coefficients and hence contains torsion information.

Stable Equivariant Construction

On the level of the based gauge quotient, the group S^1 of constant gauges acts and equivariant Seiberg-Witten Floer homologies can be defined. For example, one can follow the approach of [3], see Appendix B. We can also use equivariant singular cochains, see Appendix B. But we do not yet know whether these homologies and cohomologies are diffeomorphism invariants. (As explained before, the essential trouble is caused by reducible transition trajectories with negative spectral flow. If they are present, *in general the traditional construction of the desired chain maps breaks down completely*. Of course, this does not exclude the possibility that there might be other undiscovered schemes for constructing chain maps. *But we tend to believe that equivariant Seiberg-Witten Floer homologies are not diffeomorphism invariants*.) Instead we construct a “stable” equivariant cohomology which we show to be invariant. The word “stable” means that we couple the configuration space of the Seiberg-Witten equation with the unit circle, hence increasing the dimension of the moduli space of Seiberg-Witten points by one. In this set-up, we first obtain a stable Bott-type theory in the same way as the above Bott-type theory. Restricting to equivariant singular cochains, we then obtain the stable equivariant Seiberg-Witten Floer homology and cohomology. Note that one can also define a stable equivariant homology using differential forms analogous to the differential form construction in Appendix B, but it is not clear whether it is an invariant. Hence both in the Bott type and stable equivariant cases, *it is important to use the approach of singular chains (or similar objects) instead of differential forms*.

Spinor Perturbation

To prove the diffeomorphism invariance of the Bott-type Seiberg-Witten Floer (co)homology, we have to overcome the said obstruction of reducible transition trajectories with negative spectral flow. Our first strategy for this is to perturb the spinor equation in the transition trajectory equation in order to eliminate these transition trajectories. We utilize the vanishing of the rational homology group to construct suitable vector fields which are equivariant under based gauges. Note that *they are not equivariant under constant gauges*. A desired perturbation is then gotten by adding one of these vector fields to the spinor equation. Thus the source of our trouble, namely the vanishing of the rational homology group, also works to our benefit - a rather amusing phenomenon. There is another aspect here: in one step of the invariance proof, we have to show that the reducible trajectory from the reducible Seiberg-Witten point to itself for a fixed parameter is transversal. Here, we again use the vanishing of the rational homology group. (One can also use the Fredholm perturbations described below, which are however more complicated.)

The spinor perturbations will be applied to the stable set-up as well. Here *they are also equivariant under constant gauges*. With these perturbations, we can immediately prove the diffeomorphism invariance of the stable Bott-type theory and the stable equivariant theory.

However, for the Bott-type set-up without stabilization, a second strategy is also needed. Indeed, here the spinor perturbation will be applied whenever the spectral

flow along reducible transition trajectories from one given parameter t_1 to another t_2 is nonpositive. If the spectral flow equals 1, we perform the spinor perturbation only in one direction between t_1 and t_2 , say from t_1 to t_2 . In either case, the spectral flow in the reversed direction from t_2 to t_1 is at least 1. We need to achieve transversality in this direction as well, but here we are no longer allowed to use the spinor perturbation. The reason is that if we apply the spinor perturbation in both directions, the gauge invariance group for the glued equation from t_1 to t_1 would be too small.

Cokernel Perturbation

Our second strategy is to use Kuranishi models near reducible transition trajectories to achieve transversal perturbations which are equivariant under the full gauge group. These perturbations are given in terms of suitable operators onto the cokernel of the linearized Seiberg-Witten operator, hence we call them *cokernel perturbations*. Such perturbations were first used by Donaldson [6] in a geometric context. Note that the domain of our cokernel perturbations is the kernel of the linearized Seiberg-Witten operator. This feature depends on the positive spectral flow condition. (If the spectral flow is negative, cokernel perturbations can also be performed to achieve transversality, but the equivariance can only be preserved, provided that additional parameters are introduced. By adding additional parameters, however, we arrive at the stable set-up. The resulting perturbations are more complicated than the spinor perturbations, hence we chose to use the latter for the stable set-up.)

After performing the cokernel perturbations, the moduli space of transition trajectories between the reducibles becomes transversal. To achieve transversality for the compactified moduli spaces of transition trajectories between irreducibles or one reducible and one irreducible, we have to find a way to extend these perturbations to the situation of gluing. In other words, we have to perform cokernel perturbations near infinity, i.e. the boundary of the compactified moduli spaces. We employ extended Kuranishi models, or Kuranishi models around glued approximate transition trajectories to construct the desired perturbations. Then we can glue reducible transition trajectories with trajectories and obtain the desired transversality for the compactified moduli spaces. This important construction will be used in several steps.

Further Analysis

There are a few further delicate analytical issues in the above constructions we would like to address here. First, as explained before, we use moduli spaces of (transition) trajectories to define our boundary operator and establish the equivalence isomorphism. There is a subtlety here in the choice of the moduli spaces. Namely we need appropriate endpoint maps from the moduli spaces of trajectories to the moduli spaces of Seiberg-Witten points. For this purpose, we use the temporal model for the trajectory spaces. On the other hand, we have to show that the compactified moduli spaces have a structure of smooth manifolds with corners. It is not clear how to prove this directly for the temporal model. We work instead with another model, and introduce a "twisted time translation" action on it, which is induced from the time translation action on the temporal model. Even for this model, considerable care is needed for establishing the structure of smooth manifolds with corners.

A fundamental issue here is convergence (modulo splitting) of trajectories. When splitting occurs, we have to show the important property that the intermediate endpoints of the limit trajectories match each other. This requires a strong and detailed convergence result (Proposition 6.8).

The spinor perturbations cause additional analytical difficulties which demand special treatments. For example, one has to establish a uniform L^∞ estimate for the spinor part of the parameter-dependent Seiberg-Witten trajectories. With the presence of the perturbations, the ordinary pointwise maximum principle argument does not work. Instead, we apply the 3-dimensional Weitzenböck formula (rather than the 4-dimensional one) to obtain an initial local integral estimate in terms of the Seiberg-Witten energy. Then we apply the 4-dimensional Weitzenböck formula and the technique of Moser iteration to derive the desired L^∞ estimate.

Now we make a few concluding remarks. First, using the invariants constructed in this paper and Seiberg-Witten Floer homology for manifolds with nonzero first Betti number, one can define relative Seiberg-Witten invariants for general four dimensional manifolds with boundary. This will be discussed elsewhere. (Manifolds with first betti number equal to one requires special treatment.) Second, we would like to mention that the theory in this paper can be strengthened to a Seiberg-Witten Floer homotopy theory along the lines of the Floer homotopy theory as proposed in [5]. which amounts to the ordinary Seiberg-Witten theory with transversality replaced by Bott-type transversality. If we lift to the level of the based gauge quotient, we obtain the double Bott-type Seiberg-Witten Floer homology, which is isomorphic to the Bott-type Seiberg-Witten Floer homology. Similarly, if the manifold Y has a symmetry group, then we can consider the equivariant theory with respect to the group. Lifting it to the stable set-up, we obtain a double equivariant theory. Details will appear elsewhere.

A major part of the results in this work were obtained in Spring 1996 while both authors were at Bochum University. This work has been reported by the second named author in a number of talks given in 1996.

This work consists of two parts. The present paper is a preliminary version of Part I in which Sections 9 and 10 of Part II are also included.

2. PRELIMINARIES

To fix notations, we first recall the definitions of the Seiberg-Witten equations on 3 and 4 dimensional manifolds.

Let (X_0, g) be an oriented Riemannian manifold of dimension n and $Spin^c(X_0)$ the set of isomorphism classes of $spin^c$ structures on X_0 . Consider a $spin^c$ structure $c \in Spin^c(X_0)$ and its associated spinor bundle W and line bundle L . (More precisely, c is a representative of an element in $Spin^c(X_0)$. The homology invariants we are going to construct depend are BBBB depend are ??? independent of the choice of the representative.) We have the associated configuration space $\mathcal{A} \times \Gamma(W)$, where \mathcal{A} denotes the space of smooth unitary connections on L and Γ the space of smooth sections of a vector bundle. (We suppress the dependence on c in the notations.) The gauge transformation group (the group of gauges) is $\mathcal{G} = C^\infty(X_0, S^1)$, where $S^1 \equiv U(1)$ denotes the unit circle in \mathbb{C} . (\mathcal{G} depends only on X_0 .) The action of \mathcal{G} on $\mathcal{A} \times \Gamma(W)$ is defined by $((A, \Phi), g) \mapsto (A + g^{-1}dg, g^{-1}\Phi)$. This

formula also defines the (separate) actions of \mathcal{G} on \mathcal{A} and $\Gamma(W)$. \mathcal{G} acts freely on the subspace of pairs (A, Φ) with $\Phi \neq 0$. Such pairs are called irreducible. The isotropy subgroup at any reducible pair $(A, 0)$ is the subgroup of gauge transformations which are constants on each component of X_0 . If X_0 is connected, we identify it with S^1 . In this case, we fix a reference point $x_0 \in X_0$ and set $\mathcal{G}^0 = \{g \in \mathcal{G} : g(x_0) = 1\}$, which is called the group of based gauges. Then the quotient \mathcal{G}/S^1 is represented by \mathcal{G}^0 .

The action of a gauge g will be denoted by g^* . We set $\mathcal{B} = (\mathcal{A} \times \Gamma)/\mathcal{G}$ and $\mathcal{B}^* = (\mathcal{A} \times (\Gamma - \{0\}))/\mathcal{G}$. Let $\Omega^k(X_0)$ denote the space of smooth imaginary valued k -forms, and $\Omega^+(X_0)$ the space of smooth imaginary valued self-dual 2-forms (in the case that $\dim X_0 = 4$).

We shall need the following

Lemma 2.1. *Assume that X_0 is closed. Then the map from \mathcal{G} to $H^1(X_0; \mathbb{Z})/\{\text{torsions}\}$ given by $g \rightarrow$ the deRham class of $g^{-1}dg$ is surjective and induces an isomorphism from the component group of \mathcal{G} to $H^1(X_0; \mathbb{Z})/\{\text{torsions}\}$. Moreover, there is a unique harmonic map g with $g(x_0) = 1$ in each component of \mathcal{G} , provided that X_0 is connected and $x_0 \in X_0$ is a fixed point. In particular, \mathcal{G} is connected if X_0 is connected and $H^1(X_0; \mathbb{Z})$ is torsion.*

Proof. For simplicity, assume that X_0 is connected. The surjectivity of the said map follows from integration along paths. If $g^{-1}dg$ and $g_1^{-1}dg_1$ represent the same cohomology class, then $g_1 = ge^f$ for some $f \in \Omega^0(X_0)$ as one easily sees. Hence g_1 and g lie in the same component group.

The statement about harmonic representative follows from the standard theory of harmonic maps. It can also be derived quickly in an elementary way. For example, if $g_1 = ge^f$ and g are two harmonic maps, then f is a harmonic function, hence constant. \square

We continue with the above $spin^c$ structure c on X_0 . A connection $A \in \mathcal{A}$ induces along with the Levi-Civita connection a connection ∇^A on the spinor bundle W and the associated Dirac operator $D_A : \Gamma(W) \rightarrow \Gamma(W)$,

$$D_A = \sum_{i=1}^n e_i \cdot \nabla_{e_i}^A,$$

where $\{e_i\}$ denotes a local orthonormal tangent frame and the dot denotes the Clifford multiplication. The Dirac operator is gauge equivariant, i.e. $D_A(g^{-1}\Phi) = g^{-1}D_A\Phi$, and satisfies the following fundamental Weitzenböck formula for the Dirac operator

$$(2.1) \quad D_A^* D_A \Phi = -\Delta_A \Phi + \frac{s}{4} \Phi - \frac{1}{2} F_A \cdot \Phi,$$

where s denotes the scalar curvature of (X, g) and F_A the curvature of A .

Now we specify to the dimension $n = 4$. There is a canonical decomposition $W = W^+ \oplus W^-$ of the spinor bundle W . The Dirac operator splits: $D_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$, $D_A : \Gamma(W^-) \rightarrow \Gamma(W^+)$. For a positive spinor field $\Phi \in \Gamma(W^+)$, the curvature F_A in the above Weitzenböck formula reduces to its self dual part F_A^+

Definition 2.2. The Seiberg-Witten equation with the given $spin^c$ structure c is

$$(2.2) \quad \begin{aligned} F_A^+ &= \frac{1}{4} \langle e_i e_j \Phi, \Phi \rangle e^i \wedge e^j, \\ D_A \Phi &= 0, \end{aligned}$$

for $(A, \Phi) \in \mathcal{A} \times \Gamma(W^+)$, where $\{e^i\}$ denotes the dual of $\{e_i\}$ (a local orthonormal tangent frame). The Seiberg-Witten operator is

$$\mathbf{SW}(A, \Phi) = (F_A^+ - \frac{1}{4} \langle e_i e_j \Phi, \Phi \rangle e^i \wedge e^j, D_A \Phi).$$

Note that the Seiberg-Witten operator is gauge equivariant, where \mathcal{G} acts on 2-forms trivially. Consequently, the Seiberg-Witten equation is gauge invariant.

Next let (Y, h) be an oriented, closed Riemannian 3-manifold with metric h , and c a $spin^c$ structure on Y . We have the associated spinor bundle $S = S_c(Y)$, line bundle $L(Y) = L_c(Y)$ and the other associated spaces: $\mathcal{G}(Y)$, $\mathcal{A}(Y)$, $\mathcal{B}(Y)$, $\mathcal{B}^*(Y)$ etc. We set $X = Y \times \mathbb{R}$, which will be equipped with the product metric and given the orientation $(e_1, e_2, e_3, \frac{d}{dt})$, where (e_1, e_2, e_3) denotes a positive local orthonormal frame on Y . Let $\pi : X \rightarrow Y$ denote the projection. The $spin^c$ structure c induces a $spin^c$ structure π^*c on X with the associated line bundle $L_X = \pi^*L_Y$ and associated spinor bundles $W^+ = \pi^*S$, W^- . We have the following relation between the Clifford multiplications on S and on W^+ :

$$v \cdot \phi(y) = -(\frac{d}{dt} \cdot v \cdot \pi^* \phi)(y, 0).$$

The associated spaces for X will be indicated by the letter X . Let $i_t : Y \rightarrow X$ denote the inclusion map which sends $y \in Y$ to $(y, t) \in X$. A connection $A \in \mathcal{A}(X)$ can be written as

$$A = a(t) + f(\cdot, t)dt,$$

where $a(t) = i_t^*(A) \in \mathcal{A}(Y)$ and $f \in C^\infty(X, i\mathbb{R})$. We set $\phi(t) = \Phi(\cdot, t) = i_t^*(\Phi)$ for $\Phi \in \Gamma(W^+)$. With these notations, we can rewrite (2.1) as follows

$$(2.3) \quad \begin{aligned} \frac{\partial a}{\partial t} &= *F_a + d_Y f + \langle e_i \cdot \phi, \phi \rangle e^i, \\ \frac{\partial \phi}{\partial t} &= -\not\partial_a \phi - f\phi. \end{aligned}$$

Here F_a denotes the curvature of a , $*$ the Hodge star operator w.r.t. h , d_Y the exterior differential on Y , and $\not\partial_a$ the Dirac operator associated with the connection a .

Definition 2.3. The Seiberg-Witten energy of (A, Φ) is

$$E(A, \Phi) = \int_X (|\frac{\partial \phi}{\partial t} + f\phi|^2 + |\not\partial \phi|^2 + |\frac{\partial a}{\partial t} - df|^2 + |*F_a + \langle e_i \cdot \phi, \phi \rangle e^i|^2).$$

(The volume form is omitted.) One readily shows that it is gauge invariant.

Using the finite energy condition one easily derives from (2.3) the following limiting equation for a connection $a \in \mathcal{A}(Y)$ and a spinor field $\phi \in \Gamma(S)$

$$(2.4) \quad \begin{aligned} *F_a + \langle e_i \cdot \phi, \phi \rangle e^i &= 0, \\ \not\partial_a \phi &= 0. \end{aligned}$$

Definition 2.4. The Seiberg-Witten equation on Y with the $spin^c$ structure c is defined to be (2.4). The Seiberg-Witten operator on Y is

$$\mathbf{sw}(a, \phi) = (*F_a + \langle e_i \cdot \phi, \phi \rangle e^i, -\not\partial_a \phi).$$

As in dimension 4, the Seiberg-Witten operator \mathbf{sw} is gauge equivariant (\mathcal{G}_Y acts trivially on 1-forms).

We shall need the following perturbed Seiberg-Witten equation

$$(2.5) \quad \begin{aligned} *F_a + \langle e_i \cdot \phi, \phi \rangle e^i &= \nabla H(a), \\ \not\partial_a \phi + \lambda \phi &= 0, \end{aligned}$$

where λ denotes a real number and ∇H the L^2 -gradient of a $\mathcal{G}(Y)$ -invariant real valued function H on $\mathcal{A}(Y)$. (The L^2 -product is given in (2.7) below.) We have the associated perturbed Seiberg-Witten operator $\mathbf{sw}_{\lambda, H}$. Note that ∇H is gauge equivariant and belongs to $\ker d^*$, which are consequences of the gauge invariance of H .

The classical Chern-Simons functional plays a fundamental role in Floer's instanton homology theory. Similarly, a Chern-Simons functional associated with the 3-dimensional Seiberg-Witten equation will be important in our situation. This functional was first used by Kronheimer and Mrowka in their proof of the Thom conjecture [12].

Definition 2.5. The Chern-Simons functional with respect to a reference connection a_0 is

$$\mathbf{cs}(a, \phi) = \frac{1}{2} \int_Y (a - a_0) \wedge (F_a + F_{a_0}) + \int_Y \langle \phi, \not\partial_a \phi \rangle.$$

Let λ and H be as above. The perturbed Chern-Simons functional with perturbation (λ, H) is

$$\begin{aligned} \mathbf{cs}_{\lambda, H}(a, \phi) &= \frac{1}{2} \int_Y (a - a_0) \wedge (F_a + F_{a_0}) + \int_Y \langle \phi, \not\partial_a \phi \rangle \\ &\quad - \lambda \int_Y \langle \phi, \phi \rangle + H(a, \phi). \end{aligned}$$

Under a gauge g the perturbed Chern-Simons functional changes as follows:

$$(2.6) \quad \mathbf{cs}_{\lambda, H}(g^*(a, \phi)) = \mathbf{cs}_{(\lambda, H)}(a, \phi) + 2\pi i \int_Y c_1(L(Y)) \wedge g^{-1} dg.$$

This formula implies that $\mathbf{cs}_{\lambda, H}$ is invariant under the identity component of $\mathcal{G}(Y)$. Hence it descends to the quotient $\mathcal{B}(Y)$, provided that Y is a rational homology sphere.

We introduce an L^2 -product on $\Omega^1(Y) \oplus \Gamma(S)$:

$$(2.7) \quad \langle (\phi_1, a_1), (\phi_2, a_2) \rangle_{L^2} = \int_Y (Re \langle \phi_1, \phi_2 \rangle + \langle a_1, a_2 \rangle)$$

Here, $\langle a_1, a_2 \rangle$ denotes the (pointwise) Hermitian product. Easy computations lead to

Lemma 2.6. *The L^2 -gradient of the perturbed Chern-Simons functional is given by*

$$\nabla \mathbf{cs}_{\lambda, H}(a, \phi) = -\mathbf{sw}_{\lambda, H}.$$

It follows that the critical points of the perturbed Chern-Simons functional are precisely the solutions of the perturbed Seiberg-Witten equation.

Consider a solution (A, Φ) of the Seiberg-Witten equation on the product X . Using a suitable gauge we can transform it into temporal form. Let's assume that it is already in temporal form, i.e. $f \equiv 0$ in the formula $A = a + fdt$. Then Lemma 2.6 and the equation (2.3) imply $(\phi(t) = \Phi(\cdot, t))$

$$(2.8) \quad \frac{\partial}{\partial t}(a, \phi) = -\nabla \mathbf{cs}_{\lambda, H}(a, \phi).$$

Hence solutions of the Seiberg-Witten equation on the product X can be interpreted as trajectories (negative gradient flow lines) of the Chern-Simons functional. A similar formula and statement hold for solutions of the perturbed Seiberg-Witten equation on X , which is

$$(2.9) \quad \begin{aligned} F_A^+ &= \frac{1}{4} \langle e_i e_j \Phi, \Phi \rangle e^i \wedge e^j \\ &+ \nabla H(a) \wedge dt + *(\nabla H(a) \wedge dt), \\ D_A \Phi &= -\lambda \frac{d}{dt} \cdot \Phi, \end{aligned}$$

where $\frac{d}{dt} \cdot \Phi$ denotes the Clifford multiplication on X , and $A = a + fdt$, $\phi(t) = \Phi(\cdot, t)$ as before. The operator $\mathbf{SW}_{\lambda, H}$ is defined in an obvious way.

Next we introduce the perturbed Seiberg-Witten energy :

$$(2.10) \quad \begin{aligned} E_{\lambda, H}(A, \Phi) &= \int_X (|\frac{\partial \phi}{\partial t} + f\phi|^2 + |\not\partial \phi + \lambda \phi|^2 + |\frac{\partial a}{\partial t} - df|^2 \\ &+ |*F_a + \langle e_i \cdot \phi, \phi \rangle e^i - \nabla H(a, \phi)|^2). \end{aligned}$$

Note that it is invariant under the action of $\mathcal{G}(X)$.

Lemma 2.7. *Assume that Y is a rational homology sphere. Let $A = a + fdt$ and the gauge equivalence class of (a, ϕ) converges to $\alpha, \beta \in \mathcal{B}(Y)$ as $t \rightarrow -\infty, \infty$ respectively, then we have*

$$(2.10) \quad \begin{aligned} E_{\lambda, H}(A, \Phi) &= 2\mathbf{cs}_{\lambda, H}(\alpha) - 2\mathbf{cs}_{\lambda, H}(\beta) + \int_X |D_A \Phi|^2 \\ &+ 2 \int_X |F_A^+ - \frac{1}{4} \langle e_i e_j \Phi, \Phi \rangle e^i \wedge e^j - dt \wedge \nabla H(a) - *(dt \wedge \nabla H(a))|^2. \end{aligned}$$

In particular, there holds for a solution (A, Φ) of (2.8)

$$E_{\lambda, H}(A, \Phi) = 2\mathbf{cs}_{\lambda, H}(\alpha) - 2\mathbf{cs}_{\lambda, H}(\beta)$$

Proof. There is a similar computation in [15]. We have

$$\begin{aligned}
& \int_X |D_A \Phi|^2 + 2 \int_X |F_A^+ - \frac{1}{4} \langle e_i e_j \Phi, \Phi \rangle e^i \wedge e^j - dt \wedge \nabla H(a) - *(dt \wedge \nabla H(a))|^2 \\
&= \int_X (|\frac{\partial \phi}{\partial t} + f\phi + \not\partial \phi + \lambda \phi|^2 + |-\frac{\partial a}{\partial t} + df + *F_a + \langle e_i \cdot \phi, \phi \rangle e^i - \nabla H(a, \phi)|^2) \\
&= E(\Phi, A) + 2 \int \langle \frac{\partial \phi}{\partial t} + f\phi, \not\partial_a \phi + \lambda \phi \rangle + 2 \langle -\frac{\partial a}{\partial t} + df, *F_a + \langle e_i \cdot \phi, \phi \rangle e^i - \nabla H(a, \phi) \rangle \\
&= 2 \frac{d}{dt} \int (|\phi|^2 + \langle a - a_o, F_a \rangle) + 2H(a) + 2 \int (\langle \frac{\partial \phi}{\partial t}, \not\partial_a \phi \rangle + \langle \frac{\partial a}{\partial t} \phi, \phi \rangle).
\end{aligned}$$

Since the metric on $X = Y \times \mathbb{R}$ is the product metric, we have

$$(2.11) \quad \frac{d}{dt} \not\partial_a \phi - \not\partial_a \frac{d}{dt} \phi = \frac{\partial a}{\partial t} \phi.$$

The desired conclusion follows. \square

3. SEIBERG-WITTEN MODULI SPACES OVER Y

We continue with the (Y, h) and c of the last section. While our theory applies to arbitrary closed Y , we assume for convenience that Y is connected. Fix a reference connection a_0 . If L_Y is a trivial bundle, we choose a_0 to be the trivial connection. We have $\mathcal{A}(Y) = a_0 + \Omega^1(Y)$. We shall use the (l, p) -Sobolev norms (the $L^{l,p}$ -norms) for $l \geq 0$ and $p > 0$:

$$\|u\|_{l,p} = (\sum_{0 \leq k \leq l} \int_Y |\nabla^k u|^p)^{1/p}.$$

Consider the Sobolev spaces $\mathcal{A}_{l,p}(Y)$ and $\Gamma_{l,p}(S)$, which are the completions of $\mathcal{A}(Y)$ and $\Gamma(S)$ with respect to the (l, p) -Sobolev norm respectively. Similarly, we have the Sobolev spaces $\Omega_{l,p}^k(Y)$. The corresponding group of gauges is $\mathcal{G}_{l+1,p}(Y)$, which is the completion of $\mathcal{G}(Y)$ with respect to the $(l+1, p)$ -Sobolev norm.

We need to make a choice of the configuration spaces $\mathcal{A}_{l,p}(Y) \times \Gamma_{l,p}(S)$. We require $3p/(3-lp) > 3$ or $lp > 3$, for then all elements in $\mathcal{A}_{l,p}(Y) \times \Gamma_{l,p}(S)$ are continuous, and hence the holonomy perturbations in the sequel can be performed. Moreover, the corresponding gauges on the product space $X = Y \times \mathbb{R}$ are continuous. In particular, if we choose $l = 1$, then we require $p > 3$.

Definition 3.1. We have the following spaces of Seiberg-Witten points

$$\mathcal{SW}_{l,p} = \mathcal{SW}_{h,\lambda,H,l,p} = \{(a, \phi) \in \mathcal{A}_{l,p}(Y) \times \Gamma_{l,p}(S) : \mathbf{sw}_{\lambda,H}((a, \phi)) = 0\}$$

and the following moduli spaces of Seiberg-Witten points

$$\begin{aligned}
\mathcal{R}_{l,p} &= \mathcal{R}_{h,\lambda,H,l,p} = \mathcal{SW}_{l,p} / \mathcal{G}_{l+1,p}(Y), \\
\mathcal{R}_{l,p}^0 &= \mathcal{R}_{h,\lambda,H,l,p}^0 = \mathcal{SW}_{l,p} / \mathcal{G}_{l+1,p}^0.
\end{aligned}$$

The irreducible part of e.g. $\mathcal{R}_{l,p}$ will be denoted by $\mathcal{R}_{l,p}^*$.

We choose to work with the configuration space $\mathcal{A}_{1,4}(Y) \times \Gamma_{1,4}(Y)$. Henceforth, the subscript l stands for $(l, 4)$, e.g. $\|u\|_l = \|u\|_{l,4}$, $\mathcal{A}_l = \mathcal{A}_{l,4}$. We have the quotients $\mathcal{B}_1(Y)$ and $\mathcal{B}_1^0(Y)$ of the chosen configuration space $\mathcal{A}_1(Y) \times \Gamma_1(Y)$ under the group of gauges $\mathcal{G}_2(Y)$ and group of based gauges \mathcal{G}_2^0 . The gauge class of $(a, \phi) \in \mathcal{A}_1(Y) \times \Gamma_1(S)$ with respect to the full gauge group $\mathcal{G}_2(Y)$ will be denoted by $[a, \phi]$. Its gauge class with respect to based gauges will be denoted by $[a, \phi]^0$.

Lemma 3.2. 1) Each element in \mathcal{R}_1 or \mathcal{R}_1^0 can be represented by a smooth pair (a, ϕ) .

2) If $\|\nabla H\|_{L^\infty} < C$ for a constant C , then \mathcal{R}_1 and \mathcal{R}_1^0 are compact.

Proof. We present the proof for 2), which contains the argument for 1). Let $(a, \phi) \in \mathcal{A}_1(Y) \times \Gamma_1(S)$ be a solution of (2.5). Applying the 3-dimensional Weitzenböck formula in the weak form, the bound $\|\nabla H\|_{L^\infty} < C$ and Moser's weak maximum principle (cf. the proof of Proposition 8.5), we obtain $\|\phi\|_{L^\infty} < C$ for a constant C . (Here and in the sequel, we use the same letter C to denote all constants which appear in a priori estimation). Since $H^1(Y, \mathbb{R}) = 0$, by Hodge decomposition, $a - a_0 = d\gamma + d^*\delta$ for some $\gamma \in \Omega_0^1(Y)$ and $\delta \in \Omega_0^2(Y)$. By gauge fixing, we can assume $d^*(a - a_0) = 0$. Note that we can achieve this gauge fixing by a based gauge. Hence $a - a_0 = d^*\delta$. Furthermore, we can assume that $d\delta = 0$. Hence we have $\Delta\delta = F_a - F_{a_0}$. Since $\|F_a - F_{a_0}\| \leq C\|\phi\|^2 + C \leq C$, we have $\|\delta\|_2 \leq C$ by elliptic estimates. This implies $\|a\|_1 \leq C$. Applying this, the second equation of (2.5) and elliptic estimates, we deduce $\|\phi\|_1 \leq C$. Higher regularity and estimates follow from elliptic estimates and imply the desired compactness. \square

Henceforth we drop the subscript 1 in \mathcal{SW}_1 , \mathcal{R}_1 and \mathcal{R}_1^0 . It is clear that $\mathcal{R} = \mathcal{R}^0 / S^1$, where S^1 is the group of constant gauges. We deal with \mathcal{R}^* and \mathcal{R}^0 separately.

The moduli space \mathcal{R}^*

For a given $(a, \phi) \in \mathcal{A}_1(Y) \times \Gamma_1(Y)$, let $G_Y = G_{Y,(a,\phi)} : \Omega_1^0(Y) \rightarrow \Omega_0^1(Y) \oplus \Gamma_0(S)$ be the infinitesimal gauge action operator at (a, ϕ) , i.e. $G_Y(f) = (df, -f\phi)$. Let $G_Y^* = G_{Y,(a,\phi)}^* : \Omega_1^1(Y) \oplus \Gamma_1(S) \rightarrow \Omega_0^0(Y)$ be the formal adjoint operator of G_Y w.r.t. the inner product (2.7). We have

$$G_Y^*(b, \psi) = d^*b + \text{Im}\langle \phi, \psi \rangle.$$

There is a decomposition $\Omega_1^1(Y) \oplus \Gamma_1(S) = \ker G_Y^* \oplus \text{im } G_Y$. To be more precise, we write $\Omega_1^1(Y) \oplus \Gamma_1(S) = \ker_1 G_Y^* \oplus \text{im}_1 G_Y$. It follows that the tangent space $T_{[a,\phi]} \mathcal{B}_1^*(Y)$ of $\mathcal{B}_1^*(Y)$ at $[a, \phi]$ is represented by $\ker_1 G_{Y,(a,\phi)}^*$ (for any representative (a, ϕ) in $[a, \phi]$). Indeed, the latter gives rise to a vector bundle $\text{Ker}_1 G_Y^* \rightarrow \mathcal{A}_1 \times (\Gamma_1(S) - \{0\})$, whose quotient bundle $\text{Ker}_1 G_Y^*$ under the action of $\mathcal{G}_2(Y)$ can be identified with $T\mathcal{B}_1^*$. Next note that by the gauge invariance of the Chern-Simons functional and Lemma 2.6, we have

$$G_Y^* \mathbf{sw}_{\lambda,H}(a, \phi) = 0$$

in the weak sense for any $(a, \phi) \in \mathcal{A}_1(Y) \times \Gamma_1(Y)$. Hence the operator $\mathbf{sw}_{\lambda,H}$ defines a section $[\mathbf{sw}_{\lambda,H}]$ of the quotient bundle $\ker_0 G_Y^*$, whose fiber at $[a, \phi]$ is represented by the weak kernel $\ker_0 G_Y^*$ of \mathcal{G}_Y^* in $\Omega_0^0(Y) \oplus \Gamma_0(S)$. The moduli space \mathcal{R}_1^* is precisely the zero locus of this section.

In the sequel we omit the subscripts λ, H in the notation $\mathbf{sw}_{\lambda,H}$. Consider the operator $d\mathbf{sw}|_{(a,\phi)} : \Omega_1^1(Y) \oplus \Gamma_1(S) \rightarrow \Omega_0^1(Y) \oplus \Gamma_0(S)$, where $d\mathbf{sw}$ means the derivative, i.e. the tangent map of the operator \mathbf{sw} .) By Lemma 2.6, it is formally self-adjoint. Assume $[a, \phi] \in \mathcal{R}_1$. Then the gauge invariance of the equation $\mathbf{sw} = 0$ implies $d\mathbf{sw} \circ G_Y = 0$. It follows that $G_Y^* \circ d\mathbf{sw} = 0$. Hence we obtain an operator $d\mathbf{sw}|_{\ker_1 G_{Y,(a,\phi)}^*} : \ker_1 G_{Y,(a,\phi)}^* \rightarrow \ker_0 G_{Y,(a,\phi)}^*$ (for any representative (a, ϕ) in $[a, \phi]$). It is easy to see that this operator is Fredholm of index zero and

represents the linearization of the section $[\mathbf{sw}]$. Let it be denoted by $\mathcal{D} = \mathcal{D}_{a,\phi}$. Lemma 2.6 implies that it coincides with the Hessian operator of the Chern-Simons functional with respect to the product (2.7). Note that it extends straightforwardly to reducible Seiberg-Witten points.

Definition 3.3. *Let (a, ϕ) be a Seiberg-Witten point, i.e. a solution of the (perturbed) Seiberg-Witten equation (2.5). It is called non-degenerate if $\mathcal{D}_{a,\phi}$ is onto. The classes $[a, \phi]$ or $[a, \phi]_0$ are called nondegenerate if a representative is nondegenerate. (This is independent of the choice of the representative.)*

Lemma 3.4. *If all elements in \mathcal{R}^* are nondegenerate, then it is a naturally oriented smooth manifold of dimension zero. (The orientation means that every point in \mathcal{R}^* is assigned a sign.)*

Proof. By the above discussions, we only need to produce the natural orientation. We can use either the degree of the operator \mathbf{sw} or the spectral flow of the operator Q below as in [17]. (They give the same orientation.)

To analyse the operator \mathcal{D} , we introduce another closely related formally self-adjoint Fredholm operator Q . (The Fredholm property of \mathcal{D} is also a consequence of the Fredholm property of Q .) First notice the following deformation complex

$$(3.1) \quad 0 \longrightarrow \Omega^0 \xrightarrow{G_{a,\phi}} \Omega^1 \oplus \Gamma \xrightarrow{d\mathbf{sw}} \Omega^1 \oplus \Gamma \xrightarrow{G_{a,\phi}^*} \Omega^0 \longrightarrow 0$$

where the letters Y and S and the Sobolev subscripts are omitted in the notations. We define $Q = Q_{(a,\phi)} : (\Omega_1^1(Y) \oplus \Gamma_1(S)) \oplus \Omega_1^0(Y) \rightarrow (\Omega_0^1(Y) \oplus \Gamma_0(S)) \oplus \Omega_0^0(Y)$ by the following formula:

$$Q = \begin{pmatrix} d\mathbf{sw} & G \\ G^* & 0 \end{pmatrix}$$

Lemma 3.5. *Let $(a, \phi) \in SW$. Then we have*

$$\ker Q \cong \begin{cases} \ker \mathcal{D} \oplus \mathbb{R}, & \text{if } (a, \phi) \text{ is reducible;} \\ \ker \mathcal{D}, & \text{if } (a, \phi) \text{ is irreducible.} \end{cases}$$

and

$$\operatorname{coker} Q \cong \begin{cases} \operatorname{coker} \mathcal{D} \oplus \mathbb{R}, & \text{if } (\phi, a) \text{ is reducible;} \\ \operatorname{coker} \mathcal{D}, & \text{if } (\phi, a) \text{ is irreducible.} \end{cases}$$

We omit the simple proof.

The moduli space \mathcal{R}^0 .

The above treatment does not apply to the reducible elements of \mathcal{R} , because the tangent bundle of $\mathcal{B}_1^*(Y)$ does not extend smoothly across the reducibles. To analyse the structure of \mathcal{R}^0 around reducibles, one can use a quotient bundle formulation on the level of the based gauge quotient. But we choose a different approach which gives somewhat stronger results. Henceforth we make

Assumption 3.6. *Y is a rational homology sphere, i.e. its first Betti number is zero.*

Lemma 3.7. *There is a canonical diffeomorphism from $\Sigma \equiv (a_0 + \ker_1 d^* a) \times \Gamma_1(S)$ to $\mathcal{B}_1^0(Y)$. In other words, the former space is a global slice of the action of the group $\mathcal{G}_2^0(Y)$ on the space $\mathcal{A}_1(Y) \times \Gamma_1(S)$.*

Proof. To show that the natural map from the former space to the latter is one to one, consider $b_1, b_2 \in \ker_1 d^*$ and $\phi_1, \phi_2 \in \Gamma_1(S)$ such that $(a_0 + b_2, \phi_2) = g^*(a_0 + b_1, \phi_1)$ for some gauge $g \in \mathcal{G}_2^0$. Then $d^*(g^{-1}dg) = 0$ and $g(y_0) = 1$. Since Y is a rational homology sphere, we have $dg \equiv 0$, and hence $g \equiv 1$. The remaining part of the proof is obvious.

This lemma enables us to reduce the Seiberg-Witten operator to the said global slice. But the operator Q is no longer suitable for analysing the linearization of the Seiberg-Witten operator. Instead, we consider the following augmented Seiberg-Witten equation

$$(3.2) \quad \begin{aligned} *F_a + df + \langle e_i \cdot \phi, \phi \rangle e^i &= \nabla H(a), \\ \not\partial_a \phi + \lambda \phi + f \phi &= 0, \end{aligned}$$

where $a \in \mathcal{A}_1(Y)$, $\phi \in \Gamma_1(S)$ and $f \in \Omega_1^0(Y)$.

Lemma 3.8. *Let (a, ϕ, f) be a solution of (3.2). Then (a, ϕ) satisfies the Seiberg-Witten equation (2.5) and f is a constant. Moreover, if (a, ϕ) is irreducible, then f must be zero.*

Proof. Applying d^* to the first equation of (3.2), we deduce

$$d^*df + f|\phi|^2 = 0.$$

The desired conclusion follows. \square

We denote the left hand side of (3.2) by $\mathbf{swa}((a, \phi, f))$. The linearization of the restriction of \mathbf{swa} to Σ will be denoted by \mathcal{D}_1 . One readily checks that it is a Fredholm operator of index 1.

Definition 3.9. Let (a, ϕ) be a Seiberg-Witten point. It is called *based-nondegenerate*, if \mathcal{D}_1 is onto at (a, ϕ, f) , where f is an arbitrary constant if $\phi = 0$ and zero if $\phi \neq 0$. It is easy to see that the based-nondegenerate property is invariant under gauge transformations. In particular, this definition makes sense for based gauge classes $[a, \phi]_0$.

Let the gauges act on f trivially. The moduli space of based gauge classes of solutions of the equation (3.2) will be denoted by \mathcal{R}_a^0 . An element in it is called *based-nondegenerate*, if its corresponding element in \mathcal{R}^0 is so. As an immediate consequence of the above discussions we obtain

Lemma 3.10. *If all elements of \mathcal{R}_a^0 are based-nondegenerate, then it is a smooth oriented manifold of dimension one. If moreover \mathcal{R}^0 is compact, then the irreducible part of \mathcal{R}_a^0 consists of finitely many disjoint circles and its reducible part consists of finitely many disjoint lines. Consequently, the irreducible part of \mathcal{R}^0 consists of finitely many disjoint circles and its reducible part consists of finitely many points.*

Next we give the definition of holonomy perturbations. We follow [6] and [8]. Let D denote the unit disk in \mathbb{C} . Consider a triple (u, v, T) where $u \in Y$, $v \in T$, Y

and $I : D^2 \rightarrow Y$ is a smooth embedding such that $I(0) = y_o$ and $dI(T_0 D)$ is transversal to v_0 . Fix a point $s_0 \in S^1$. Let $P(y_0, v_0, I)$ be the set of all smooth embeddings $\gamma : S^1 \times D \rightarrow Y$ such that $\gamma(s_0, \theta) = I(\theta)$ for $\theta \in D$ and $\frac{\partial \gamma}{\partial s}(s_0, 0) = v_0$. Here s_0 is a fixed point in S^1 . We set

$$P^{(m)} = \cup_{(y_o, v_0, I)} (P(y_0, v_o, I))^m,$$

for $m \in \mathbb{N}$. Now we define a map $\gamma^h : \mathcal{A}(Y) \times P^{(m)} \rightarrow C^\infty(D^2, U(1)^m)$ by

$$\gamma^h(a, (\gamma^1, \gamma^2, \dots, \gamma^m))(\theta) = (\gamma_\theta^1(a), \gamma_\theta^2(a), \dots, \gamma_\theta^m(a)),$$

where $\gamma_\theta^i : \mathcal{A}(Y) \rightarrow U(1)$ denotes the holonomy map along the loop $\gamma^i(\cdot, \theta)$ (at the base point y_0). It is easy to see that γ^h is gauge invariant. Next we choose a sequence $\{\epsilon_i\}$ of positive numbers as in [8] such that

$$C^\epsilon(U(1)^m, \mathbb{R}) = \{u \in C^\infty(U(1)^m, \mathbb{R}) : \|u\|_\epsilon < \infty\}$$

is complete. Here

$$\|u\|_\epsilon = \sum_{i=0}^{\infty} \epsilon_i \max_{U(1)^m} |\nabla^i u|.$$

Now we set

$$\Pi = \cup_{m \in \mathbb{N}} (P_m \times C^\epsilon(U(1)^m, \mathbb{R})).$$

This is the parameter space of holonomy perturbations. Choose a smooth function ξ with support in the interior of D . For each $\pi = (\gamma, u) \in \Pi$, we define the holonomy perturbation $H_\pi : \mathcal{A}(Y) \rightarrow \mathbb{R}$ by

$$H_\pi(a) = \int_{D^2} u(\gamma^h(a)) \xi(\theta) d^2 \theta.$$

It is clear that H_π extends to $\mathcal{A}_1(Y)$.

Lemma 3.11. *For any $\pi = (\gamma, u) \in \Pi$, H_π is a smooth $\mathcal{G}_2(Y)$ -invariant function. Moreover, the L^2 -gradient ∇H_π satisfies*

$$\|\nabla H_\pi(a)\|_{L^\infty} \leq C,$$

with $C > 0$ independent of $a \in \mathcal{A}$. Similar bounds hold for the higher derivatives of H_π . The bounds can be made arbitrarily small by choosing u small.

Proof. For simplicity, we only consider $m = 1$. Set $H = H_\pi$. We can write $H(a) = \int_{D^2} u(\gamma_\theta(a)) \xi d^2 \theta$. It follows that

$$dH(a)(b) = \int_{D^2} du|_{\gamma_\theta(a)} (\gamma'_\theta(a)b) \xi d^2 \theta.$$

Elementary computations lead to

$$\gamma'_\theta(a)b = -\gamma_\theta(a) \int b.$$

We deduce

$$\begin{aligned}
dH(a)(b) &= - \int_{D^2} \xi \langle \nabla u, \gamma_\theta(a) \rangle d^2\theta \int_{\gamma_\theta} b \\
&= \int_{\gamma(S^1 \times D^2)} (\xi \circ \gamma^{-1}) \langle b, \overline{\langle \nabla u, \gamma_\theta(a) \rangle} (\gamma^{-1})^*(dt) \rangle \left| \frac{\partial \gamma}{\partial t} \right|^{-2} (\gamma^{-1})^*(dtd^2\theta), \\
&= \int_Y f(\xi \circ \gamma^{-1}) \langle b, \overline{\langle \nabla u, \gamma_\theta(a) \rangle} (\gamma^{-1})^*(dt) \rangle
\end{aligned}$$

where

$$f = \left| \frac{\partial \gamma}{\partial t} \right|^{-2} |(\gamma^{-1})^*(dtd^2\theta)/d\text{vol}|.$$

Consequently, $\nabla H(a) = f(\xi \circ \gamma^{-1}) \overline{\langle \nabla u, \gamma_\theta(a) \rangle} (\gamma^{-1})^*(dt)$. The desired estimate for ∇H follows. The higher order derivatives can easily be computed by using the above formula. \square

We make

Assumption 3.12. *Henceforth we choose H in (2.5) and (3.2) to be H_π .*

We remark in passing that for the purpose of achieving transversality for the moduli spaces \mathcal{R}^* and \mathcal{R}^0 it is not necessary to introduce the holonomy perturbations. However, they are important for achieving transversality for Seiberg-Witten trajectories as will be seen in the next section.

Lemma 3.13. *For perturbation $\pi \in \Pi$ such that $\nabla^2 H_\pi$ is small enough in L^∞ -norm (the set of such π is a nonempty open set), there exists a unique reducible element $[a, 0] \in \mathcal{R}$. Equivalently, there is a unique $a \in \mathcal{A}_1(Y)$ such that*

$$\begin{aligned}
(3.3) \quad *F_a - \nabla H_\pi(a) &= 0, \\
d^*a &= 0.
\end{aligned}$$

Proof. Since Y is a rational homology sphere, the operator $*d : \ker d^* \rightarrow \ker d^*$ is a bounded isomorphism. Hence the existence follows from the implicit function theorem. To prove the uniqueness, consider connections a and a_1 satisfying (3.3). We set $b = a - a_1$ and deduce

$$*db = \nabla H_\pi(a_1) - \nabla H_\pi(a) \text{ and } d^*b = 0.$$

By the implicit function theorem, for π with the property stated in the lemma, $b = 0$. \square

The unique solution of (3.3) will be denoted by $a_{(h,\pi)}$.

Lemma 3.14. *For $\pi \in \Pi$ satisfying the condition of Lemma 3.13, let $\sigma(\mathcal{D}_{a_{(h,\pi)}})$ be the set of eigenvalues of $\mathcal{D}_{a_{(h,\pi)}}$. Assume that $\nabla^3 H_\pi$ is small enough in L^∞ -norm (the set of such π is a nonempty open set). Then for $\lambda \in (\mathbb{R} - \sigma(\mathcal{D}_{a_{(h,\pi)}}))$, all elements of $\mathcal{R}(Y)$ are nondegenerate and all elements in $\mathcal{R}^0(Y)$ are based-nondegenerate.*

Proof. We only present the proof for the statement concerning the non-degeneracy. The based nondegeneracy can be treated in a similar way. Consider $(a, \psi) \in [\mathcal{A}, \psi] \subset$

$\mathcal{R}(Y)$, we are going to show that \mathcal{D} at (a, ϕ) is onto. By gauge equivariance, we can choose $a = a_{(h, \pi)}$ for the reducible element. By Lemma 3.5, it suffices to analyse the operator Q . We have

$$Q(b, \psi, f) = \begin{pmatrix} *db + 2\langle e_i \cdot \phi, \psi \rangle e^i + df - \nabla^2 H(a)b \\ \not\partial_a \psi + \lambda \psi + f\phi + b\phi \\ d^*b + \text{Im} \langle \phi, \psi \rangle \end{pmatrix}.$$

Consider an element $(b_1, \psi_1, f_1) \in \Omega_0^1(Y) \oplus \Gamma_0(S) \oplus \Omega_0^0(Y)$ satisfying

$$(3.4) \quad \langle Q(b, \psi, f), (b_1, \psi_1, f_1) \rangle_{L^2} = 0$$

for all $(b, \psi, f) \in \Omega_1^1(Y) \oplus \Gamma_1(S) \oplus \Omega_1^0(Y)$. We first derive that (b_1, ψ_1, f_1) satisfies the adjoint equation $Q^* = 0$ (hence it satisfies $Q = 0$ because $Q^* = Q$) and is smooth.

Case 1 $\phi = 0$ and $a = a_{(h, \pi)}$.

We have $\not\partial_a \psi_1 + \lambda \psi_1 = 0$. By the choice of λ , we conclude that $\psi_1 \equiv 0$. Now (b_1, f_1) satisfies the following equation

$$(3.5) \quad \begin{aligned} *db_1 + df_1 - \nabla^2 H(a)b_1 &= 0, \\ d^*b_1 &= 0. \end{aligned}$$

Since Y is a rational homology sphere, the operator $(b_1, f_1) \rightarrow (*db_1 + df_1, d^*b_1)$ is an isomorphism from $\Omega_1^1(Y) \oplus (\Omega_1^0(Y))^0$ onto $\Omega_0^1(Y) \oplus (\Omega_0^0(Y))^0$, where the superscript 0 means the condition that the average be zero. As in the proof of Lemma 3.9, we deduce that if π has been chosen small enough, f_1 must be a constant and $b_1 = 0$. We conclude that $\text{coker } Q \cong \mathbb{R}$. By Lemma 3.5, this implies that \mathcal{D} is onto.

Case 2 $\phi \neq 0$.

By the unique continuation, the set $U = \{\phi \neq 0\}$ is an open dense set. For $y \in U$, $e_1 \cdot \phi(y), e_2 \cdot \phi(y), e_3 \cdot \phi(y)$ and $\phi(y)$ span S_y , where S_y denotes the fiber of S at $y \in Y$. We deduce that $\psi_1(y) = 0$ for $y \in U$, whence $\psi_1 \equiv 0$.

Now we easily see that (b_1, f_1) satisfies the equation (3.5). Hence $b_1 \equiv 0$ and the equation $Q(b_1, \psi_1, f_1) = 0$ reduces to $f_1\phi = 0$. It follows that $f_1 = 0$ in U and consequently $f_1 \equiv 0$. We conclude that Q is onto. By Lemma 3.5, \mathcal{D} is onto. \square

As a consequence of the previous lemmas, we deduce

Proposition 3.15. *Let π and λ satisfy the same conditions as in Lemma 3.13 and Lemma 3.14. Then \mathcal{R} consists of finitely many signed points, the irreducible part of \mathcal{R}^0 consists of finitely many disjoint oriented circles, and its reducible part is a signed point.*

Definition 3.16. We call π and λ “good”, provided that they satisfy the conditions of Lemma 3.13 and Lemma 3.14. The set of good parameters (π, λ) is a nonempty open set.

4. SEIBERG-WITTEN TRAJECTORIES: TRANSVERSALITY

By Seiberg-Witten trajectories we mean solutions of the equation (2.9). (We shall choose $H = H_\pi$.) As stated in the introduction, our goal is to establish a Morse Floer theory for the Chern-Simons functional on the quotient space $\mathcal{B}^0(Y)$.

$(\mathcal{A}_1(Y) \times \Gamma_1(Y))/\mathcal{G}_2^0(Y)$. The union of the critical submanifolds of the Chern-Simons functional is precisely the moduli space \mathcal{R}^0 . The negative gradient flow lines of the Chern-Simons functional are given by the temporal form of the Seiberg-Witten trajectories, cf. Section 2. Setting $A = a(t) + f(\cdot, t)dt$ and $\phi(t) = \Phi(\cdot, t)$ in (2.9) we can rewrite it as follows

$$(4.1) \quad \begin{aligned} \frac{\partial a}{\partial t} - *F_a - d_Y f - \langle e_i \cdot \phi, \phi \rangle e^i &= \nabla H(a), \\ \frac{\partial \phi}{\partial t} + \not{D}_a \phi + \lambda \phi + f \phi &= 0. \end{aligned}$$

(We omit the subscript π in H_π .)

We shall use various spaces of local $(l, 4)$ Sobolev class ($L_{loc}^{l,4}$ class), e.g. $\mathcal{A}_{l,loc} = \mathcal{A}_{l,4,loc}(X)$, $\Gamma_{l,4,loc}^\pm = \Gamma_{l,4,loc}(W^\pm)$, $\Omega_{l,loc}^k = \Omega_{l,4,loc}^k(X)$ and $\mathcal{G}_{l,loc}(X) = L_{loc}^{l,4}(X, S^1)$.

Definition 4.1. *We have the following spaces of Seiberg-Witten trajectories:*

$\mathcal{N} = \{(A, \Phi) \in \mathcal{A}_{1,loc} \times \Gamma_{1,loc}^+ : (A, \Phi) \text{ solves (4.1) and has finite (perturbed) Seiberg-Witten energy}\}.$

The corresponding moduli spaces are:

$$\mathcal{M} = \mathcal{N}/\mathcal{G}_{2,loc}(X) \text{ and } \mathcal{M}^0 = \mathcal{N}/\mathcal{G}_{2,loc}^0(X),$$

where $\mathcal{G}_{2,loc}^0(X) = \{g \in \mathcal{G}_{2,loc}(X) : g((y_0, 0)) = 1\}$. (Recall that y_0 is a fixed reference point in Y .)

Proposition 4.2. *Assume that λ and π are good. Then there are positive constants C and ε_0 depending only on h , λ and π with the following properties. For any temporal Seiberg-Witten trajectory $u = (A, \Phi) = (\phi, a)$ of local $(1, 4)$ -Sobolev class and finite energy, there exist a gauge $g \in \mathcal{G}_2(Y)$ and two smooth solutions u_- , u_+ of (2.5) such that g^*u is smooth and the following holds. For all l , $\|g^*u(\cdot, t) - u_\pm\|_l < C(l)e^{-C|t|}$ for $|t| \geq T$, where $C(l)$ depends only on h , λ , π , l and an upper bound of the energy of u , and $T > 0$ satisfies $E(u, \{|t| > T\}) \leq \varepsilon_0$. ($E(u, \Omega)$ means the energy of u on the domain Ω .) Moreover, if u_- and u_+ are not gauge equivalent, then*

$$E(u) \geq \varepsilon_0.$$

The proof will be given in Part II.

Corollary 4.3. *We have*

$$\mathcal{M} = \cup_{\alpha, \beta \in \mathcal{R}} \mathcal{M}(\alpha, \beta), \mathcal{M}^0 = \cup_{\alpha, \beta \in \mathcal{R}} \mathcal{M}^0(\alpha, \beta),$$

where $\mathcal{M}(\alpha, \beta) = \mathcal{N}(\alpha, \beta)/\mathcal{G}_{2,loc}(X)$ and $\mathcal{M}^0(\alpha, \beta) = \mathcal{N}(\alpha, \beta)/\mathcal{G}_{2,loc}^0(X)$ and for sets B_1 and B_2 of Seiberg-Witten points,

$$(4.2) \quad \begin{aligned} \mathcal{N}(B_1, B_2) &= \{u \in \mathcal{N} : \text{there is a } g \in \mathcal{G}_{2,loc}(X) \text{ such that} \\ g^*u(\cdot, t) &\text{ converges exponentially to some } p \in B_1 \text{ as } t \rightarrow -\infty \\ &\text{and to some } q \in B_2 \text{ as } t \rightarrow +\infty\}. \end{aligned}$$

We shall use the moduli spaces $\mathcal{M}^0(\alpha, \beta)$ (or the corresponding moduli spaces of trajectories in temporal form) to construct the boundary operator in our Bott type chain (cochain) complex.

Proposition 4.2 and its corollary suggest that we can work in the set-up of exponentially converging trajectories. We introduce various relevant spaces. For a positive function ξ on X we consider the following ξ -weighted $(l, 4)$ -Sobolev norms

$$(4.3) \quad \|u\|_{l,\xi} = \left(\sum_{k \leq l} \int_X \xi |\nabla^k u|^4 dy dt \right)^{\frac{1}{4}}.$$

For each pair of nonnegative numbers $\delta = (\delta_-, \delta_+)$ we choose a positive smooth function δ_F on \mathbb{R} such that $\delta_F(t) = \delta_{\pm}|t|$ near $\pm\infty$. We have the following δ -weighted $(l, 4)$ -Sobolev norms ($L_{\delta}^{l,4}$ -norms)

$$\|u\|_{l,\delta} = \|u\|_{l,e^{\delta_F}}.$$

Let $\Omega_{l,\delta}^k = \Omega_{l,\delta}^k(X)$, $\mathcal{A}_{l,\delta} = \mathcal{A}_{l,\delta}(X)$ and $\Gamma_{l,\delta}^{\pm} = \Gamma_{l,\delta}(W^{\pm})$ denote the completion of the obvious spaces w.r.t. the $L_{\delta}^{l,4}$ -norm.

For $u \in \mathcal{A}_1(Y) \times \Gamma_1(S)$, let G_u denote its isotropy group of gauge actions. (It is trivial if u is irreducible and S^1 if u is reducible.) The isotropy groups are identical for gauge equivalent elements, hence $G_{[u]}$ is well-defined.

In the following definition, the notation \mathcal{G}_{δ} should not be confused with e.g. \mathcal{G}_2 . To avoid inconsistency, we require that δ_+ and δ_- be smaller than 1.

Definition 4.4. For $p, q \in \mathcal{A}_1(Y) \times \Gamma_1(S)$ we introduce

$$L_{\delta}(p, q) = \{u \in \mathcal{A}_{1,loc} \times \Gamma_{1,loc}^+ : u - u_0 \in \Omega_{1,\delta}^1 \times \Gamma_{1,\delta}^+ \text{ for some } u_0 \in \mathcal{A}_{1,loc} \times \Gamma_{1,loc}^+ \text{ which near } \infty(-\infty) \text{ is } t\text{-dependent and equals } p(q)\}.$$

$$\mathcal{G}_{\delta}(p, q) = \{g \in \mathcal{G}_{2,loc}(X) : g - g_0 \in L_{\delta}^{2,4}(X, \mathbb{C}) \text{ for some } g_0 \in \mathcal{G}_{2,loc} \text{ which is } t\text{-independent and belongs to } G_p(G_q) \text{ near } \infty(-\infty)\},$$

$$\mathcal{G}_{\delta}^0(p, q) = \{g \in \mathcal{G}_{\delta}(p, q) : g(y_0, 0) = 1\},$$

$$\mathcal{G}_{\delta}^I = \{g \in \mathcal{G}_{2,loc}(X) : g - 1 \in L_{\delta}^{2,4}(X, \mathbb{C})\},$$

$$\mathcal{G}_{\delta}^{I,0} = \{g \in \mathcal{G}_{\delta}^I : g(y_0, 0) = 1\}.$$

Note that the second and third groups act freely. The first acts freely on the irreducible part of $L_{\delta}(p, q)$. We have the quotients: $\mathcal{B}_{\delta} = \mathcal{B}_{\delta}(p, q) = L_{\delta}(p, q)/\mathcal{G}_{\delta}$, $\mathcal{B}_{\delta}^0 = L_{\delta}/\mathcal{G}_{\delta}^0$,

$$\text{and } \mathcal{B}_{\delta}^I = L_{\delta}/\mathcal{G}_{\delta}^I.$$

Because $\mathcal{G}_2(Y)$ is connected, we obtain equivalent (in terms of suitable gauges) spaces for Seiberg-Witten points p', q' which are gauge equivalent to p, q respectively.

Definition 4.5. For $\alpha, \beta \in \mathcal{B}_1(Y)$ we introduce

$$L_{\delta}(\alpha, \beta) = \cup_{p \in \alpha, q \in \beta} L_{\delta}(p, q),$$

$$\mathcal{G}_{\delta}^{\infty} = \{g \in \mathcal{G}_{2,loc}(X) : g - g_0 \in L_{\delta}^{2,4}(X, \mathbb{C}) \text{ for some } g_0 \in L_{loc}^{2,4} \text{ which is } t\text{-independent near } \pm\infty\},$$

$$\mathcal{G}_{\delta}^{\infty,0} = \{g \in \mathcal{G}_{\delta}^{\infty} : g(y_0, 0) = 1\}.$$

Definition 4.6. For $p, q \in SW$, we set $\mathcal{N}_{\delta}(p, q) = L_{\delta}(p, q) \cap \mathcal{N}$. We have the moduli spaces of trajectories $\mathcal{M}_{\delta}(p, q) = \mathcal{N}_{\delta}(p, q)/\mathcal{G}_{\delta}(p, q)$, $\mathcal{M}_{\delta}^0(p, q) = \mathcal{N}_{\delta}(p, q)/\mathcal{G}_{\delta}^0(p, q)$ and $\mathcal{M}_{\delta}^I(p, q) = \mathcal{N}_{\delta}(p, q)/\mathcal{G}_{\delta}^I$.

Definition 4.7. For $\alpha, \beta \in \mathcal{R}$, we set $\mathcal{N}_\delta(\alpha, \beta) = L_\delta(\alpha, \beta) \cap \mathcal{N}$. We have the moduli spaces:

$$\mathcal{M}_\delta(\alpha, \beta) = \mathcal{N}_\delta(\alpha, \beta) / \mathcal{G}_\delta^\infty, \quad \mathcal{M}_\delta^0(\alpha, \beta) = \mathcal{N}_\delta(\alpha, \beta) / \mathcal{G}_\delta^{\infty, 0}.$$

The irreducible part of e.g. \mathcal{M}_δ will be denoted by \mathcal{M}_δ^* . The spaces $\mathcal{M}_\delta(p, q)$ and $\mathcal{M}_\delta^0(p, q)$ are canonically isomorphic to $\mathcal{M}_\delta([p], [q])$ and $\mathcal{M}_\delta^0([p], [q])$ respectively. Moreover, we have the following easy lemma.

Lemma 4.8. If (π, λ) is good, p, q are smooth representatives of α and β respectively, and δ_-, δ_+ are positive and less than the exponent c in Proposition 4.3, then there are canonical isomorphisms

$$\mathcal{M}_\delta(\alpha, \beta) \cong \mathcal{M}(\alpha, \beta), \quad \mathcal{M}_\delta^0(\alpha, \beta) \cong \mathcal{M}^0(\alpha, \beta).$$

We use these isomorphisms to topologize $\mathcal{M}(\alpha, \beta)$ and $\mathcal{M}^0(\alpha, \beta)$.

Thus, $\mathcal{M}(\alpha, \beta)$, $\mathcal{M}_\delta(\alpha, \beta)$ and $\mathcal{M}_\delta(p, q)$ can be viewed as three different models of the same space. The same holds for the spaces $\mathcal{M}^0(\alpha, \beta)$ etc.. To analyse the structures of these moduli spaces, we focus on the set-up $\mathcal{B}_\delta^I(p, q)$ and $\mathcal{M}_\delta^I(p, q)$.

For $p, q \in \mathcal{A}_1(Y) \times \Gamma_1(S)$, consider the infinitesimal gauge action operator $G_X = G_{X, (A, \Phi)} : \Omega_{1, \delta}^0 \rightarrow \Omega_{0, \delta}^1 \oplus \Gamma_{0, \delta}^+$ at a given $(A, \Phi) \in L_\delta(p, q)$, $G_X(f) = (df, -f\Phi)$. Let G_X^* be the formal adjoint operator of G_X w.r.t. the following inner product

$$(4.4) \quad \langle (\Phi_1, A_1), (\Phi_2, A_2) \rangle_\delta = \int_X (Re \langle \Phi_1, \Phi_2 \rangle + \langle A_1, A_2 \rangle) e^{\delta_F} dydt.$$

We have the following elementary lemma, which is analogous to Proposition 2a.1 in [8].

Lemma 4.9. If δ_- and δ_+ are positive and small enough, then $\Omega_{1, \delta}^1 \oplus \Gamma_{1, \delta}^+ = im_1 G_X \oplus ker_1 G_X^*$. Consequently, the tangent space $T_{[A, \Phi]} \mathcal{B}_\delta(p, q)$ is represented by $ker G_X^*$.

Now we assume that δ satisfies the condition of Lemma 4.8. Let $\mathcal{U}_\delta \rightarrow \mathcal{B}_\delta^I$ denote the quotient bundle of the trivial bundle $L_\delta(p, q) \times (\Omega_{1, \delta}^+ \oplus \Gamma_{1, \delta}^-) \rightarrow L_\delta(p, q)$ ($\mathcal{G}_{2, \delta}$ acts on Ω^+ trivially and on Γ^- by $g^* \Psi = g^{-1} \Psi$). The (perturbed) Seiberg-Witten operator $\mathbf{SW} = \mathbf{SW}_{\lambda, H}$ (cf. Section 2) induces a section $[\mathbf{SW}]$ of this bundle. If $p, q \in \mathcal{SW}$, then its zero locus is precisely the moduli space $\mathcal{M}_\delta(p, q)$.

The linearization of the section $[\mathbf{SW}]$ is given by the restriction of the operator $d\mathbf{SW}|_{(A, \Phi)}$ to $ker_1 G_X$, which will be denoted by $\mathcal{D}_X = \mathcal{D}_{X, (A, \Phi)}$. We introduce another closely related operator $\mathcal{F}_{p, q} : \Omega_{1, \delta}^1 \oplus \Gamma_{1, \delta}^+ \rightarrow \Omega_{0, \delta}^1 \oplus \Gamma_{0, \delta}^- \oplus \Omega_{0, \delta}^0$

$$\mathcal{F}_{p, q} = \begin{pmatrix} d\mathbf{SW} \\ G_X^* \end{pmatrix}.$$

Lemma 4.10. If $p, q \in \mathcal{SW}$, $(A, \Phi) \in \mathcal{N}_\delta(p, q)$, then $ker \mathcal{D}_X = ker \mathcal{F}_{p, q}$ and $coker \mathcal{D}_X = coker \mathcal{F}_{p, q}$. Consequently, \mathcal{D}_X is Fredholm iff $\mathcal{F}_{p, q}$ is Fredholm. If they are Fredholm, they have the same index.

Proof. The kernel equality is clear. By the gauge invariance of the Seiberg-Witten equation, we have $d\mathbf{SW} \circ G_X = 0$. Applying this and Lemma 4.7, BBBB Lemma 4.7 is not right BBBB we derive $im \mathcal{F}_{p, q} = im \mathcal{D}_X \oplus G_X^*(im_1 G_X)$. BBBB I am not sure, pls check BBBB But the second summand equals $\Omega_{1, \delta}^0$. \square

Lemma 4.11. *Assume that δ_- and d_+ are small enough. If p (q) is reducible, we assume in addition that δ_- (δ_+) is positive. Then $\mathcal{F}_{p,q}$ is Fredholm for all $p, q \in \mathcal{SW}$.*

Proof. Let $\Omega^{k,Y}$ denote the subspace of Ω^k consisting of forms which do not contain dt . Then $\Omega_{1,\delta}^+$ can be identified with $\Omega_\delta^{1,Y}$. Using this identification, we have for a given $(A, \Phi) = (a + fdt, \phi)$

$$\mathcal{F}_{p,q}(b + \tilde{f}dt, \psi) = \frac{d}{dt} \begin{pmatrix} b \\ \psi \\ \tilde{f} \end{pmatrix} - \begin{pmatrix} *d_Y b + d_Y \tilde{f} + 2\langle e_i \phi, \psi \rangle e^i - \nabla^2 H(a) \cdot b \\ -\partial_a \psi - \lambda \phi - b\phi - f\psi - \tilde{f}\phi \\ d_Y^* b - \delta'_F \tilde{f} + \text{Im}\langle \phi, \psi \rangle \end{pmatrix}.$$

Hence $\mathcal{F}_{p,q} - (\frac{d}{dt} - Q + (0, 0, \delta'_F)) = (0, f, 0)$, where Q was defined in Section 3. Since f decays exponentially, its multiplication is a compact operator. Now the limits of the operator $Q - (0, 0, \delta'_F)$ at $\pm\infty$ are formally self-adjoint. Hence we can follow [8] or [16] to show that $\frac{d}{dt} + Q - \delta'_F$ is Fredholm. Consequently, $\mathcal{F}_{p,q}$ is Fredholm. \square

Consider a good pair (π_0, λ) . Choose a neighborhood Π_0 of π_0 such that $\Pi_0 \times \{\lambda\}$ consists of good pairs and the smallness conditions in Lemma 4.11 is uniform for all $\pi \in \Pi_0$ (with λ fixed).

Proposition 4.12. *Assume that δ_- and δ_+ are positive, satisfy the above smallness condition for all $\pi \in \Pi_0$ and are less than the constant c in Proposition 4.2. Then for generic $\pi \in \Pi_0$ the following holds. For all $p, q \in \mathcal{SW}$, $[\mathbf{SW}]$ is transversal to the zero section, and hence \mathcal{M}_δ^I is a smooth manifold of dimension $\text{ind } \mathcal{F}_{p,q}$. Consequently, \mathcal{M}_δ is a smooth manifold of dimension*

$$\text{ind } \mathcal{F}_{p,q} - \dim G_p - \dim G_q$$

and \mathcal{M}_δ^0 is a smooth manifold of dimension

$$\text{ind } \mathcal{F}_{p,q} - \dim G_p - \dim G_q + 1.$$

Proof. First assume that at least one of p, q is irreducible. Then all elements of $L_\delta(p, q)$ are irreducible. We extend the bundle $\mathcal{U}_\delta \rightarrow \mathcal{B}_\delta^I \times \Pi_0$ in the trivial way. Then $[\mathbf{SW}]$ gives rise to a section of the extended bundle. By the Sard-Smale theorem, it suffices to show that this section is transversal to the zero section, which amounts to the surjectivity of the operator $\mathcal{D}_X \oplus d_\pi \mathbf{SW}$ at all (A, Φ) which solve the Seiberg-Witten equation with parameter $\pi \in \Pi_0$ (and λ). By Lemma 4.8, the latter is equivalent to the surjectivity of the operator $\mathcal{F}_{p,q} \oplus d_\pi \mathbf{SW}$, which in turn follows from the spinor part of the transversality argument in [12] (which is similar to the argument in the proof of Lemma 3.14) and Floer's transversality argument in [8] based on holonomy perturbations. This establishes the statement about the space \mathcal{M}_δ^I . The statements about \mathcal{M}_δ^0 and \mathcal{M}_δ^* follow via the involved group actions.

If both p and q are reducible, then they represent the same (unique) reducible element in \mathcal{R} . By Lemma 2.7, the energy of every Seiberg-Witten trajectory equals zero. By the gauge equivariance of the operator $\mathcal{F}_{p,q}$, we can use temporal gauges and assume that $a = a_0 = (a_0, 0)$ and $(A, \Phi) = a_0$. The asserted transversality

follows from Lemma C.1 in Appendix C. Both \mathcal{M}_δ^I and \mathcal{M}_δ^0 are circles in this case. \square

Note that by gauge equivariance, the transversality property is independent of the choice of the representatives p, q in their gauge classes. Since \mathcal{R} consists of finitely many points, for generic π , the transversality property is shared by all p, q with $[p], [q] \in \mathcal{R}$.

Definition 4.13. We shall say that those π and the corresponding λ as described above are *generic*.

By gauge equivariance, $d\mathbf{SW}, G_X$ and G_X^* are equivariant as can easily be verified. By this and the homotopy invariance of Fredholm index, we also have

Lemma 4.14. *For given $\alpha, \beta \in \mathcal{R}$, $\text{ind } \mathcal{F}_{p,q}$ is independent of the choice of $p \in \alpha, q \in \beta$ and $(A, \Phi) \in L_\delta(p, q)$. It is also independent of the choice of (δ_-, δ_+) (satisfying the smallness condition in Proposition 4.12).*

Remark 4.15 We have derived the transversality of the space \mathcal{M}_δ^0 in terms of the transversality of \mathcal{M}_δ^I . We present another, more direct argument, which will be useful for analysing the moduli spaces of parameter-dependent trajectories. Let $\mathcal{B}_\delta^{I,0}$ denote the quotient of $L_\delta(p, q)$ under the action of the group $\mathcal{G}_\delta^{I,0}$, and $\mathcal{M}_\delta^{I,0}$ denote the quotient of $\mathcal{N}_\delta(p, q)$ under the action of the same group. (The element in $\mathcal{B}_\delta^{I,0}$ determined by (A, Φ) will be denoted by $[A, \Phi]_0$.) We need to derive the transversality of this later moduli space. There is a quotient bundle \mathcal{U}_δ^0 over $\mathcal{B}_\delta^{I,0}$ which is analogous to the bundle \mathcal{U}_δ . As before, the operator \mathbf{SW} induces a section $[\mathbf{SW}]_0$ of the bundle \mathcal{U}_δ^0 , whose zero locus is precisely the moduli space $\mathcal{M}_\delta^{I,0}$.

Choose a smooth imaginary valued function f_0 on X with $f_0(y_0, 0) = \sqrt{-1}$. If both the limits p and q are irreducible, we choose f_0 to be compactly supported. Otherwise, we choose f_0 to take the value $\sqrt{-1}$ near the infinity which corresponds to the reducible limit. We obtain a decomposition $\Omega_{1,\delta}^1 \oplus \Gamma_{1,\delta}^+ = \text{im}_1^0 G_X \oplus \ker_1 G_X^* \oplus \text{span } \{G_X f_0\}$, where im_1^0 means the image of those f with $f(y_0, 0) = 0$. (The third factor is not orthogonal to the first one.) It follows that the tangent space $T_{[A,\Phi]_0} \mathcal{B}_\delta^{I,0}$ is represented by $\ker_1 G_X^* \oplus \text{span } \{G_X f_0\}$. Let P_{f_0} denote the orthogonal projection to the orthogonal complement of $G_X^* G_X f_0$ (with respect to the product (4.2)). Then there holds $\ker_1 G_X^* \oplus \text{span } \{G_X f_0\} = \ker P_{f_0} G_X^*$. Now the linearization of the section $[\mathbf{SW}]_0$ is given by the restriction of the derivative $d\mathbf{SW}$ (at (A, Φ)) to $\ker P_{f_0} G_X^*$, which we denote by \mathcal{D}_X^0 . There is an operator $\mathcal{F}_{p,q}^0$ analogous to $\mathcal{F}_{p,q}$ which is the combination of $d\mathbf{SW}$ with $P_{f_0} G_X^*$. A statement about the relation between \mathcal{D}_X^0 and $\mathcal{F}_{p,q}^0$ similar to Lemma 4.10 holds. In particular, they have the same index. The transversality argument in the proof of Proposition 4.11 also applies straightforwardly to $\mathcal{F}_{p,q}^0$. Since the moduli space \mathcal{M}_δ^0 is the quotient of $\mathcal{M}_\delta^{I,0}$ under the free action of a compact Lie group, the transversality of the former follows. This Lie group is trivial if both p and q are irreducible, and the circle if at least one of them is reducible.

We note the following relation:

$$(4.3) \quad \text{ind } \mathcal{F}_{p,q}^0 = \text{ind } \mathcal{F}_{p,q} + 1.$$

Finally, we state an important consequence of transversality.

Lemma 4.16. *Let (π, λ) be generic, $p, q \in \mathcal{SW}$ and $u \in \mathcal{N}_\delta(p, q)$. Choose a reference $u_0 \in L_\delta(p, q)$. Then $d\mathbf{SW}_u$ has a right inverse $Q_u : \Omega_{0,\delta}^+ \oplus \Gamma_{0,\delta}^- \rightarrow \Omega_{1,\delta}^+ \oplus \Gamma_{1,\delta}^+$ with*

$$\|Q_u\| \leq C,$$

where C depends only on $\|u - u_0\|_{1,\delta}$ and (π, λ) . Q_u is equivariant under gauge actions. In particular, $\|Q_{g^*u}\| \leq \|g\|_{C^1} \|Q_u\|$ for $g \in \mathcal{G}_\delta^\infty \cap C^1(X, S^1)$.

Proof. We proceed in the context of the above remark. Let \mathcal{O}_u denote the L_δ^2 -orthogonal complement of $\ker d\mathbf{SW}_u$ in $\ker P_{f_0} G_X^*$. Then the operator $d\mathbf{SW}_u|_{\mathcal{O}_u}$ is a bounded isomorphism onto $\Omega_{0,\delta}^+ \oplus \Gamma_{0,\delta}^-$. We define Q_u to be its inverse. The stated norm estimates and gauge equivariance follow readily. \square

5. INDEX AND ORIENTATION

Consider a good pair (π, λ) . Let $O = O_{h,\pi}$ be the unique reducible element in \mathcal{R} . For $\alpha \in \mathcal{R}$ we define

$$\mu(\alpha) = \text{ind } \mathcal{F}_{p,q} - 1,$$

where $p \in \alpha, q \in O$.

Note that μ can easily be extended to all elements of $\mathcal{B}_1(Y)$. It depends on h, π and λ . Elementary computation shows $\mu(O) = 0$.

Lemma 5.1. *For $p, q, r \in \mathcal{SW}$ there holds*

$$\text{ind } \mathcal{F}_{p,r} = \text{ind } \mathcal{F}_{p,q} + \text{ind } \mathcal{F}_{q,r} - \dim G_q.$$

An analogous formula holds for the operator $\mathcal{F}_{p,r}^0$.

Proof. This is similar to the corresponding index addition formula in Floer's theory [8]. Floer's argument can be applied directly. Another argument is as follows. Composing with weight multiplication operators, we can transform the operators to Sobolev spaces without weight. Then the addition formula is the consequence of a linear version of the gluing argument in Part II. The term $\dim G_q$ arises because of the “jumping” across the kernel of the operator $d^* + \text{Im}\langle \phi, \rangle$ which is caused by the operator $(0, 0, \delta'_F)$.

Corollary 5.2. *There holds*

$$\text{ind } \mathcal{F}_{p,q} = \mu([p]) - \mu([q]) + \dim G_q.$$

Consequently, if (π, λ) is a generic pair, then we have $\dim \mathcal{M}_\delta(p, q) = \mu([p]) - \mu([q]) - \dim G_p$, $\dim \mathcal{M}_\delta^0(p, q) = \mu([p]) - \mu([q]) + 1 - \dim G_p$.

Next we study the orientation of the moduli spaces of Seiberg-Witten trajectories.

Proposition 5.3. *Assume that (π, λ) is generic. Then $\mathcal{M}_\delta^I(p, q), \mathcal{M}_\delta(p, q)$ and $\mathcal{M}_\delta^0(p, q)$ are orientable. Indeed, their orientations are canonically determined after some choices are made, which will be given in the proof below. (Consequently, $\mathcal{M}_\delta([p], [q]), \mathcal{M}_\delta^0([p], [q]), \mathcal{M}([p], [q])$ and $\mathcal{M}^0([p], [q])$ are orientable.) Moreover, the orientations are consistent with the gluing construction used in the proof below.*

namely the orientation of $\mathcal{M}_\delta^I(p, q)$ is the same as the product orientation induced from gluing $\mathcal{M}_\delta^I(p, r)$ to $\mathcal{M}_\delta^I(r, q)$.

Proof. Without loss of generality, assume that δ_+ and δ_- are equal and positive. The operator $\mathcal{F}_{p,q}$ induces a section of Fredholm operators over \mathcal{B}_δ^I . Let $\det(p, q) = \det \mathcal{F}_{p,q}$ be its determinant line bundle. Indeed, the operator section $\mathcal{F}_{p,q}$ can be deformed through Fredholm operator sections to the operator section $(d^+ + d_\delta^*, D_A)$, whose determinant line bundle is trivial. Hence $\det(p, q)$ is trivial. An orientation of the vector space $H_\delta^0 \oplus H_\delta^1 \oplus H_\delta^+$ (the homology of the complex associated with the operator $d^+ + d_\delta^*$) then determines an orientation of $\det(p, q)$, cf. [19].

To obtain consistent orientations, we choose a smooth irreducible pair $p_0 = (a_0, \phi_0)$ and consider the space $L_\delta(p_0, p)$ and its quotient $\mathcal{B}_\delta^I(p_0, p)$ for $p \in \mathcal{SW}$. By the proof of Lemma 4.11, the operator section $\mathcal{F}_{p_0,p}$ is a Fredholm operator section. By the above argument, its determinant line $\det(p_0, p)$ is trivial. We fix an orientation (a trivialization) for it. For $p, q \in \mathcal{SW}$ we construct an embedding by a simple gluing process

$$L_\delta(p_0, p) \times L_\delta(p, q) \rightarrow L_\delta(p_0, q).$$

(Compare [10].) On the other hand, we choose reference elements $u_0 \in L_\delta(p_0, p)$ and $u_1 \in L_\delta(p, q)$. Then it is easy to show that $u_0, u_1 + (\ker d_\delta^* \times \Gamma_{1,\delta}^+)$ are global slices in $L_\delta(p_0, p)$ and $L_\delta(p, q)$ for the action of the groups \mathcal{G}^I respectively. Using them and the above embedding we obtain an embedding Θ :

$$\mathcal{B}_\delta^I(p_0, p) \times \mathcal{B}_\delta^I(p, q) \rightarrow \mathcal{B}_1(p_0, q).$$

We have the projections π_0 , and π_1 of the above product to its factors. In addition let π_p be its projection to p . Now the index addition formula (Lemma 5.1) leads to an addition formula for the index bundle, which in turn implies a product formula for the determinant line bundle. We apply the last formula to the present situation to deduce

$$\pi_0^* \det(p_0, p) \otimes \pi_1^* \det(p, q) \otimes \pi_p^* l_p \cong \det(p_0, q) |_{\text{im } \Theta},$$

where l_p is the dual of the kernel of the operator $d^* + \langle \phi, \cdot \rangle$ at p . We choose an orientation for l_O (note that the l_p 's are canonically equivalent to each other for $p \in O$). Then the above isomorphism determines an orientation of $\det(p, q)$, which gives rise to an orientation of $\mathcal{M}_\delta^I(p, q)$. The desired consistency follows from the construction.

If both p and q are reducible, then \mathcal{M}_δ^0 is a circle generated by gauge actions, and hence inherits a canonical orientation from the actions. Otherwise, the moduli space \mathcal{M}_δ is the quotient of \mathcal{M}_δ^I by a free S^1 action if one of p, q is reducible, and equals \mathcal{M}_δ^I if neither is reducible. Hence the orientation of \mathcal{M}_δ^I induces an orientation of \mathcal{M}_δ . On the other hand, \mathcal{M}_δ is the quotient of \mathcal{M}_δ^0 by another free S^1 action, hence we arrive at an orientation of \mathcal{M}_δ^0 . \square

Remark 5.4. *As the transversality of $\mathcal{M}_\delta^0(p, q)$ can be derived directly through the study of the operator $\mathcal{F}_{p,q}^0$, so can its orientability. The orientability part of the above arguments can be modified to suit the situation of $\mathcal{M}^0(p, q)$. But we need a different arrangement for consistent orientation here, which will be discussed in Section 7*

6. THE TEMPORAL MODEL AND COMPACTIFICATION

A temporal Seiberg-Witten trajectory is a solution (A, Φ) of (4.1) which is temporal (in temporal form, or “in temporal gauge”), i.e. $A = a + fdt$ with $f \equiv 0$. For $B_1, B_2 \subset \mathcal{SW}$ we set

$$\mathcal{N}_T(B_1, B_2) = \{u \in \mathcal{N}(B_1, B_2) : u \text{ is temporal}\},$$

$$\mathcal{N}_{T,\delta}(B_1, B_2) = \{u \in \mathcal{N}_\delta(B_1, B_2) : u \text{ is temporal}\}.$$

For $\alpha \in \mathcal{R}$ let S_α denote its lift to \mathcal{R}^0 . For $\alpha, \beta \in \mathcal{R}$ we set

$$\mathcal{M}_T(S_\alpha, S_\beta) = \mathcal{N}_T(S_\alpha, S_\beta) / \mathcal{G}_2^0(Y),$$

$$\mathcal{M}_{T,\delta}(S_\alpha, S_\beta) = \mathcal{N}_{T,\delta}(S_\alpha, S_\beta) / \mathcal{G}_2^0(Y).$$

By Proposition 4.2, for good parameters and small δ , the first two spaces are identical, and the last two are so, too. We shall only consider good parameters and small δ . Note that we can also assume that the space $\mathcal{M}(p, q)$ is identical to the space $\mathcal{M}_\delta(p, q)$.

Lemma 6.1. *Assume that (π, λ) is generic. Then there are canonical diffeomorphisms from $\mathcal{M}^0(p, q)$ and $\mathcal{M}^0(\alpha, \beta)$ to $\mathcal{M}_T(S_\alpha, S_\beta)$ for any $p \in \alpha, q \in \beta$, where $\alpha, \beta \in \mathcal{R}$. (Indeed, these diffeomorphisms prove the manifold structure of $\mathcal{M}_T(S_\alpha, S_\beta)$.)*

Proof. Let $L_{\delta,T}(\alpha, \beta)$ denote the temporal part of $L_\delta(\alpha, \beta)$, which is a smooth submanifold as can easily be seen. We have the following temporal transformation T_G : for $u = (a + fdt, \Phi)$, set

$$g_T(u) = e^{-\int_0^t f}$$

and $T_G(u) = g_T(u)^*u$. Clearly, $T_G : \mathcal{M}^0(\alpha, \beta) \rightarrow L_{\delta,T}(\alpha, \beta) / \mathcal{G}_2^0(Y)$ is a smooth map with image $\mathcal{M}_T(S_\alpha, S_\beta)$. One readily checks that this map is an embedding. \square

We need to compactify our moduli spaces of trajectories. First we introduce some terminology.

Definition 6.2. 1. Let $p, q \in \mathcal{SW}$. A k -trajectory $u = (u_m)_{1 \leq m \leq k}$ from p to q with consecutive junctures $p_0 = p, \dots, p_k = q \in \mathcal{SW}$ is an element in $\mathcal{N}(p_0, p_1) \times \mathcal{N}(p_{k-1}, p_k)$ with $u_m \in \mathcal{N}(p_{m-1}, p_m)$. u^m is called the m -th portion of u . p, q are called the endpoints of u at $+\infty$ and $-\infty$ respectively. u is called proper, if its junctures belong to distinct gauge classes with respect to the full gauge group.

Definition 6.3. For $u \in \mathcal{N}(p, q)$, let p', q' be the endpoints of $T_G(u)$ at $-\infty$ and $+\infty$ respectively. We set $\pi^-(u) = [p']_0, \pi_+(u) = [q']_0$. π_+ and π_- are called the temporal endpoint maps.

Definition 6.4. A k -trajectory (u_m) is called consistent, if $\pi_+(u_m) = \pi_-(u_{m+1})$ for all $1 \leq m \leq k-1$. For distinct $p_0, \dots, p_k \in \mathcal{SW}$, let $\mathcal{N}(p_0, \dots, p_k)$ denote the space of consistent (and proper) k -trajectories with junctures p_0, \dots, p_k . For $p, q \in \mathcal{SW}_0, p \neq q$, let $\mathcal{N}(p, q)^k$ denote the space of proper and consistent k -trajectories from p to q . We set $\hat{\mathcal{N}}(p, q) = \cup_k \mathcal{N}(p, q)^k$. Note that the temporal endpoints maps π_+ and π_- naturally extend to consistent k -trajectories.

Definition 6.5. Let e.g. $\hat{\mathcal{N}}_T(p, q)$ denote the subspace of $\hat{\mathcal{N}}_T(p, q)$ consisting of temporal k -trajectories. For $\alpha, \beta \in \mathcal{R}$ we set $\hat{\mathcal{N}}_T(S_\alpha, S_\beta) = \cup_{p \in \gamma \in S_\alpha, q \in \gamma' \in S_\beta} \hat{\mathcal{N}}_T(p, q)$ and

$$\hat{\mathcal{M}}_T(S_\alpha, S_\beta) = \hat{\mathcal{N}}_T(S_\alpha, S_\beta) / \mathcal{G}_2^0(Y).$$

(The action occurs on each portion of k -trajectories.) We also have subspaces $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)_k$ of $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$, which provide it with a natural stratification. Furthermore, we have subspaces $\hat{\mathcal{M}}_T(S_{\alpha_0}, \dots, S_{\alpha_k})$ for distinct $\alpha_0, \dots, \alpha_k \in \mathcal{R}$.

Definition 6.6. For each $\alpha \in \mathcal{R}$ we choose an element $p_\alpha \in \alpha$. We fix this choice henceforth and denote the set of these elements by \mathcal{SW}_0 . Let $\mathcal{N}(p, q; \mathcal{SW}_0)^k$ denote the subspace of $\mathcal{N}(p, q)$ consisting of k -trajectories whose junctures belong to \mathcal{SW}_0 . Similarly, we have $\mathcal{N}(p, q; \mathcal{SW}_0)$.

For distinct $p_0, \dots, p_k \in \mathcal{SW}$, we set

$$\mathcal{M}^0(p_0, \dots, p_k) = \mathcal{N}(p_0, \dots, p_k) / (\mathcal{G}_\delta^0(p_0, p_1) \times \dots \times \mathcal{G}_\delta^0(p_{k-1}, p_k)).$$

For $p, q \in \mathcal{SW}_0$, let $\mathcal{M}^0(p, q)^k$ denote the union of all $\mathcal{M}^0(p_0, \dots, p_k)$ with distinct $p_0 = p, \dots, p_k = q \in \mathcal{SW}_0$, and let $\hat{\mathcal{M}}^0(p, q)$ denote the union of $\hat{\mathcal{M}}^0(p, q)^k$ over all possible k . By the definition, $\hat{\mathcal{M}}^0(p, q)$ has a natural stratification.

Lemma 6.7. Assume that (π, λ) is generic. Then $\mathcal{M}^0(p_0, \dots, p_k)$ and $\mathcal{M}_T(S_{\alpha_0}, \dots, S_{\alpha_k})$ are canonically diffeomorphic smooth manifolds, where $\alpha_0, \dots, \alpha_k \in \mathcal{R}$ are distinct and $p_i \in \alpha_i$.

Proposition 6.8. Assume that π, λ are good. Then there is a k_0 depending only on h, π and λ such that the following holds. Let u^j be a sequence of temporal Seiberg-Witten trajectories with uniformly bounded energy. Then there is a sequence of gauges $g_j \in \mathcal{G}_2^0(Y)$ with the following properties:

1. Set $\tilde{u}^j = g_j^* u^j$. Then each \tilde{u}^j is smooth and converges exponentially at time infinities as described in Proposition 4.2. Let p_j and q_j denote its limits at $+\infty$ and $-\infty$ respectively.

2. After passing to a subsequence of u^j , p_j converge to a p , and q_j converge to a q . Moreover, under the assumption that p differs from q , \tilde{u}^j converge to a proper k -trajectory $u = (u_m) \in \hat{\mathcal{N}}_T(p, q)^k$ with $k \leq k_0$ in the following sense:

- i) $E(\tilde{u}^j) \rightarrow E(u) \equiv \sum_m E(u_m)$,
- ii) there are decompositions $\mathbb{R} = \cup_{1 \leq m \leq k} [t_m^j, t_{m+1}^j]$ with $t_m^j \in [-\infty, +\infty]$, $t_{m+1}^j > t_m^j$ and $t_{m+1}^j - t_m^j \rightarrow +\infty$ as $j \rightarrow \infty$, such that for each $l \geq 0$,

$$(6.1) \quad \sup_{Y \times [t_m^j - T_m^j, t_{m+1}^j - T_m^j]} e^{\delta_F} |\nabla^l u^j(\cdot + T_m^j, \cdot) - \nabla^l u_m| \rightarrow 0$$

as $j \rightarrow \infty$, where T_m^j are constants with the property $t_m^j - T_m^j \rightarrow -\infty$ and $t_{m+1}^j - T_m^j \rightarrow +\infty$ as $j \rightarrow \infty$, and the δ_+, δ_- in δ are positive and less than the exponent C in Proposition 4.2.

(Note that i) is actually implied by ii.)

In short, the above proposition says that each \tilde{u}^j splits into k portions to yield k new sequences, which converge in exponentially weighted norms after suitable time translation adjustments. The fact that the total limit is a k -trajectory implies in particular that adjacent portions converge to trajectories whose endpoints match. This is an important property.

Corollary. *Let $u^j \in \mathcal{N}(p, q)$ with $p, q \in \mathcal{SW}_0, p \neq q$. Then there are gauges $g_j \in \mathcal{G}_{2,\delta}^0(p, q)$ such that $g_j^* u^j$ are smooth and after passing to a subsequence, they converge to a proper and consistent k -trajectory $u \in \hat{\mathcal{N}}(p, q; \mathcal{SW}_0)$ with $k \leq k_0$ in the sense as described in the proposition.*

The proof of this proposition will be given in Part II.

We extend the concept of convergence of trajectories to k -trajectories to convergence of k -trajectories to k' -trajectories in the obvious way. Then Proposition 6.8 and its corollary extend straightforwardly to k -trajectories.

The real line \mathbb{R} acts on trajectories in terms of the time translation. We define the time translation action on k -trajectories to be the separate time translation action on each portion of k -trajectories. It gives rise to an action of \mathbb{R}^k . Let the underline denote quotient under the time translation action in the context of temporal trajectories, e.g. $\underline{\mathcal{M}}_T(S_\alpha, S_\beta) = \mathcal{M}_T(S_\alpha, S_\beta)/\mathbb{R}$.

Proposition 6.9. *Assume the same as in Proposition 6.8. Then the spaces $\hat{\underline{\mathcal{M}}}_T(S_\alpha, S_\beta)$ are compact, where the topology is given by the convergence concept in Proposition 6.4 and its corollary. Similarly, for $p, q \in \mathcal{SW}_0$, the quotient of $\hat{\underline{\mathcal{M}}}^0(p, q)$ under the time translation action is compact.*

Consider $p \in \gamma \in S_\alpha, q \in \gamma' \in S_\beta$. The spaces $\hat{\mathcal{M}}^0(p, q)$ (or $\mathcal{M}^0(p, q)$) and $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$ (or $\mathcal{M}_T(S_\alpha, S_\beta)$) are isomorphic. But their quotients under the time translation action are not isomorphic. For our purpose, the time translation action on the temporal model is more suitable. For this reason, we consider the action of \mathbb{R} on $\mathcal{N}(p, q)$ induced from the time translation action on $\mathcal{N}_T(S_\alpha, S_\beta)$, which we call the “twisted time translation”.

Definition 6.10. For $R \in \mathbb{R}$, let τ_R denote the time translation by R , i.e.

$$\tau_R(u)(y, t) = u(y, t - R).$$

The twisted time translation T_R by R is defined as follows. Let $u \in \mathcal{N}(p, q)$. Then $T_R u = (g_T(u)^{-1})^*(\tau_R(T_G(u)))$. (See the proof of Lemma 6.1 for g_T and T_G .) The twisted time translation acts on each portion of k -trajectories separately, giving rise to a \mathbb{R}^k action. We use the underline to denote the quotient under the twisted time translation action.

Lemma 6.11. *Assume the same as in Proposition 6.8. Then $\underline{\mathcal{M}}^0(p_0, \dots, p_k)$ and $\underline{\mathcal{M}}_T(S_{\alpha_0}, \dots, S_{\alpha_k})$ are canonically diffeomorphic smooth manifolds, where p_i, α_i are the same as in Lemma 6.7.*

The following proposition is a consequence of Proposition 6.4. We assume $p, q \in \mathcal{SW}_0, p \neq q$.

Proposition 6.12. *Assume the same as in Proposition 6.8. Then the spaces $\hat{\underline{\mathcal{M}}}^0(p, q)$ are compact. More precisely, consider e.g. a sequence $[u^j]_0 \in \mathcal{M}^0(p, q)$. There is a sequence $v^j \in \mathcal{N}(p, q)$ such that*

- 1 $T_{c_j} v^j \in [u^j]_0$ for some c_j .
- 2 after passing to a subsequence, v^j converges to a proper and consistent k -trajectory $v = (v^m) \in \hat{\mathcal{N}}(p, q; \mathcal{SW}_0)$.

Notice that the compact space $\hat{\underline{\mathcal{M}}}_T(S_\alpha, S_\beta)$ contains $\underline{\mathcal{M}}_T(S_\alpha, S_\beta)$ as a subspace. The compactification of $\mathcal{M}_T(S_\alpha, S_\beta)$ is given by its closure $\hat{\underline{\mathcal{M}}}_T(S_\alpha, S_\beta)$.

Proposition 6.13. *In a generic situation, we have $\bar{\mathcal{M}}_T(S_\alpha, S_\beta) = \hat{\mathcal{M}}_T(S_\alpha, S_\beta)$. Moreover, the following hold:*

(1) $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$ has the structure of d -dimensional smooth orientable manifolds with corners (i.e. modeled on the first quadrant of \mathbb{R}^d), where $d = \mu(\alpha) - \mu(\alpha) - \dim G_p + 1$ with $[p] \in \alpha, [q] \in \beta$.

(2) This structure is compatible with the stratification $\hat{\mathcal{M}}_T(S_\alpha, S_\beta) = \cup_k \hat{\mathcal{M}}_T(S_\alpha, S_\beta)^k$, i.e. the interior of the k -dimensional edge of $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$ is exactly $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)^k$.

(3) The temporal endpoint maps $\pi_- : \hat{\mathcal{M}}_T(S_\alpha, S_\beta) \rightarrow S_\alpha$ and $\pi_+ : \hat{\mathcal{M}}_T(S_\alpha, S_\beta) \rightarrow S_\beta$ are smooth fibrations. (They are naturally induced from the previous temporal endpoint maps.)

In particular, we have

$$(6.2) \quad \begin{aligned} \partial \hat{\mathcal{M}}_T(S_\alpha, S_\beta) &= \cup_{\mu(\alpha) > \mu(\gamma) > \mu(\beta)} \hat{\mathcal{M}}_T(S_\alpha, S_\gamma, S_\beta) \\ &= \cup_{\mu(\alpha) > \mu(\gamma) > \mu(\beta)} \hat{\mathcal{M}}_T(S_\alpha, S_\gamma) \times_{S_\gamma} \hat{\mathcal{M}}_T(S_\gamma, S_\beta), \end{aligned}$$

where the fiber product space $\hat{\mathcal{M}}_T(S_\alpha, S_\gamma) \times_{S_\gamma} \hat{\mathcal{M}}_T(S_\gamma, S_\beta)$ is defined to be

$$(6.3) \quad \{(u, v) \in \hat{\mathcal{M}}_T(S_\alpha, S_\gamma) \times \hat{\mathcal{M}}_T(S_\gamma, S_\beta) : \pi_+(u) = \pi_-(v)\}.$$

The following is an equivalent formulation for Proposition 6.13. (We only give part of the statements.)

Proposition 6.14. *In a generic situation, we have $\bar{\mathcal{M}}^0(p, q) = \hat{\mathcal{M}}^0(p, q)$. Moreover, $\hat{\mathcal{M}}^0(p, q)$ has the structure of d -dimensional manifolds with corners which is compatible with its natural stratification.*

The said equivalence means the following:

Lemma 6.15. *In a generic situation, $\bar{\mathcal{M}}^0(p, q)$ is canonically diffeomorphic to $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$.*

This lemma is an easy consequence of the temporal transformation once Proposition 6.14 has been established. Hence Proposition 6.13 can be derived as a corollary of Proposition 6.14. It may be possible to prove it without appealing to Proposition 6.14, but that seems rather cumbersome. Note that Proposition 6.14 alone would suffice for our purpose, but Proposition 6.13 is important from a conceptual point of view.

Proposition 6.14 is a consequence of the compactness result Proposition 6.7 and a result on structures near infinity which we present now.

We only consider generic situations. Let u be a Seiberg-Witten trajectory of finite energy. We define $\rho_+(u)$ and $\rho_-(u)$ by the following equations

$$E(u, \{t \geq \rho_+\}) = \varepsilon, E(u, \{t \leq \rho_-\}) = \varepsilon,$$

where ε is given in Proposition 4.2. Let $\langle u \rangle$ denote the set of trajectories which are gotten from u by a twisted time translation. Let $u_+^*(u)$ denote the element in $\langle u \rangle$ whose ρ_+ value equals zero, and $u_-^*(u)$ denote that with the ρ_- value equaling zero. By a real analyticity argument, one readily shows that u_+^* and u_-^* give rise to transversal slices for the twisted time translation action. We set $\mathcal{N}^*(p, q) = \{u \in \mathcal{N}(p, q) : \rho_-(u) = 0\}$. Similarly, we have $\mathcal{N}^*(p, q)^k$ etc..

For each u , there are unique numbers $R_+(u)$ and $R_-(u)$ such that $u = T_{R_+(u)}(u_+^*)$ and $u = T_{R_-(u)}(u_-^*)$. In particular, $\mathcal{N}(p, q)^k$ is isomorphic to $\mathcal{N}^*(p, q) \times \mathbb{R}^k$.

Proposition 6.16. *There are neighborhoods U , \hat{U} of $\underline{\mathcal{M}}^0(p, q)^k$ in $\underline{\mathcal{M}}^0(p, q)^k \times [0, \infty)^{k-1}$ and $\hat{\mathcal{M}}(p, q)$ respectively, and a homeomorphism $\mathbf{F} : U \rightarrow \hat{U}$ such that the restriction of \mathbf{F} to $U_0 = U \cap (\underline{\mathcal{M}}^0(p, q)^k \times (0, \infty)^{k-1})$ is a diffeomorphism. Moreover, the following hold:*

1) *For each compact set K in $\underline{\mathcal{M}}^0(p, q)^k$, there is a positive number r_0 such that $K \times [0, r_0]^{k-1} \subset U$.*

2) *For $1 \leq j \leq k-1$, the restriction of \mathbf{F} to the j -th boundary stratum of U is a diffeomorphism onto the j -th boundary stratum $\hat{U} \cap \underline{\mathcal{M}}^0(p, q)^{j+1}$ of $U \cap \underline{\mathcal{M}}^0(p, q)$. Here e.g. the first boundary stratum of $\underline{\mathcal{M}}^0(p, q)^k \times [0, \infty)^{k-1}$ is $\underline{\mathcal{M}}^0(p, q)^k \times (\{0\} \times (0, \infty)^{k-2} \cup (0, \infty) \times \{0\} \times (0, \infty)^{k-2} \dots \cup (0, \infty)^{k-2} \times \{0\})$.*

\mathbf{F} defines the structure of smooth manifolds with corners for $\hat{\mathcal{M}}^0(p, q)$ stated in Proposition 6.13.

The proof of this proposition will be given in Part II.

7. BOTT-TYPE AND STABLE EQUIVARIANT HOMOLOGY

We first introduce a few orientation conventions. We follow those used in [9]. For an oriented smooth manifold with corners \mathcal{X} , its boundary is oriented in such a way that $\text{span}\{n_{\partial\mathcal{X}}\} \oplus T\partial\mathcal{X} = TX|_{\partial\mathcal{X}}$ as oriented vector bundles (away from the corners of $\partial\mathcal{X}$), where $n_{\partial\mathcal{X}}$ is an inward normal field of the boundary. Given transversal smooth maps $F_1 : \mathcal{X}_1 \rightarrow S$ and $F_2 : \mathcal{X}_2 \rightarrow S$ from two oriented smooth manifolds with corners into an oriented smooth manifold S , the fiber product $\mathcal{X}_1 \times_S \mathcal{X}_2 = (F_1 \times F_2)^{-1}(\text{Diag}(S \times S))$ (Diag means *diagonal*) has a canonical orientation such that $T(\mathcal{X}_1 \times_S \mathcal{X}_2) \oplus N = (-1)^{\dim S \cdot \dim \mathcal{X}_2} TX_1 \oplus TX_2$ as oriented bundles, where N denotes the oriented bundle $(d(F_1 \times F_2))^{-1}((TS \oplus \{0\})|_{\text{Diag}(S \times S)})$. The following lemma is due to Fukaya and easy to verify.

Lemma 7.1. *There hold*

$$(7.1) \quad \partial(\mathcal{X}_1 \times_S \mathcal{X}_2) = \partial\mathcal{X}_1 \times_S \mathcal{X}_2 + (-1)^{\dim \mathcal{X}_1 + \dim S} \mathcal{X}_1 \times_S \partial\mathcal{X}_2,$$

and

$$(7.2) \quad (X_1 \times_S \mathcal{X}_2) \times_{S'} \mathcal{X}_3 = X_1 \times_S (\mathcal{X}_2 \times_{S'} \mathcal{X}_3).$$

We shall use the natural orientations of S_α ($\alpha \in \mathcal{R}$) provided by Lemma 3.10. (We can also use any other orientations.) By Proposition 6.13, the boundary of $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$ is a union of fiber products. We need to arrange the orientation of these spaces so that a suitable consistency holds in regard of the natural orientation of fiber products as defined above and boundary orientations. Indeed, we have

Lemma 7.2. *We can choose the orientation of $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$ such that*

$$(7.3) \quad \partial \hat{\mathcal{M}}_T(S_\alpha, S_\beta) = (-1)^{\mu(\alpha) + \dim S_\alpha} \bigcup_{\mu(\alpha) > \mu(\gamma) > \mu(\beta)} \hat{\mathcal{M}}_T(S_\alpha, S_\gamma) \times_{S_\gamma} \hat{\mathcal{M}}_T(S_\gamma, S_\beta)$$

as oriented manifolds.

This lemma is analogous to Sublemma 1.20 in [9] and can be proven by the same arguments as there.

Bott-type Seiberg-Witten Floer homology

For a topological space \mathcal{X} , let $C_j(\mathcal{X})$ denote the free abelian group of singular j -chains in \mathcal{X} with coefficient \mathbb{Z} . Let $A_j(\mathcal{X})$ be its subgroup generated by elements of the form $(\Delta_j, f) - (\Delta_j, f')$, where Δ_j denotes the standard Euclidean j -simplex and f, f' are homotopic continuous maps from Δ_j to \mathcal{X} . Let $Br_j(\mathcal{X})$ denote its subgroup generated by elements of the form $(\Delta_j, f) - \sigma$, where σ is obtained from (Δ_j, f) by a baricentric subdivision. We set

$$\tilde{C}_j(\mathcal{X}) = C_j(\mathcal{X}) / (A_j(\mathcal{X}) + Br_j(\mathcal{X}))$$

and define $\tilde{C}^j(\mathcal{X})$ to be the dual of $\tilde{C}_j(\mathcal{X})$, i.e. the free abelian group of homomorphisms from $\tilde{C}_j(\mathcal{X})$ to \mathbb{Z} . For $\sigma \in C_j(\mathcal{X})$, its equivalence class in $\tilde{C}_j(\mathcal{X})$ will be denoted by $\langle \sigma \rangle$. Let ∂_O denote the ordinary boundary operator on $C_j(\mathcal{X})$.

Remark 7.3. For $\sigma = (\Delta_j, f) \in C_j(\mathcal{X})$, the map $(\partial\Delta_j, f|_{\partial\Delta_j})$ from the oriented boundary induces a chain in a natural way. By our convention for boundary orientation, this chain equals $-\partial\sigma$.

Next we set $S_i = \cup\{S_\alpha : \alpha \in \mathcal{R}, \mu(\alpha) = i\}$ and

$$\tilde{C}_k = \oplus_{i+j=k} \tilde{C}_j(S_i), \tilde{C}^k = \oplus_{i+j=k} \tilde{C}^j(S_i),$$

$$\tilde{C}_* = \oplus_k \tilde{C}_k, \tilde{C}^* = \oplus_k \tilde{C}^k.$$

We proceed to define a boundary operator $\tilde{\partial} : \tilde{C}_k \rightarrow \tilde{C}_{k-1}$ with the dual (coboundary operator) $\tilde{\partial}^* : \tilde{C}^{k-1} \rightarrow \tilde{C}^k$ for each k . First, for each pair $\alpha, \beta \in \mathcal{R}$ with $\mu(\alpha) > \mu(\beta)$ we define a boundary operator $\partial_{\alpha, \beta} : C_k \rightarrow C_{k-1}$, where

$$C_k = \oplus_{i+j=k} C_j(S_i).$$

If the moduli space $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$ is empty, we define $\partial_{\alpha, \beta}$ to be the zero operator. If it is nonempty, we define $\partial_{\alpha, \beta}$ as follows. For $\sigma \notin C_*(S_\alpha)$, we set $\partial_{\alpha, \beta}\sigma = 0$. For $\sigma = (\Delta_j, f) \in C_j(S_\alpha)$ with $\mu(\alpha) + j = k$, consider the fiber product

$$\Delta = \Delta_j \times_{S_\alpha} \mathcal{M}_T(S_\alpha, S_\beta) = \{(z, u) \in \Delta_j \times \mathcal{M}_T(S_\alpha, S_\beta) : f(z) = \pi_-(u)\}.$$

We have a natural map $\tilde{\pi}_+ : \Delta \rightarrow S_\beta$, $\tilde{\pi}_+((z, u)) = \pi_+(u)$. The fiber product Δ is a $(j + \mu(\alpha) - \mu(\beta) - 1)$ -dimensional compact oriented manifold with corners, hence can be triangulated into oriented simplices which are identified with Euclidean simplices. (We choose such a triangulation and identifications arbitrarily.) The map $\tilde{\pi}_+ : \Delta \rightarrow S_\beta$ then gives rise to a singular $(j + \mu(\alpha) - \mu(\beta) - 1)$ -chain in S_β . We define this chain to be $\partial_{\alpha, \beta}\sigma$. Clearly, we indeed have

$$\partial_{\alpha, \beta} : C_k \rightarrow C_{k-1}$$

for all k .

Definition 7.4. We define $\partial : C_k \rightarrow C_{k-1}$ as follows. First, we define $\partial_0 : C_k \rightarrow C_{k-1}$ by $\partial_0 = (-1)^k \partial_O$. Then we set $\partial = \partial_0 + \sum_{\mu(\alpha) > \mu(\beta)} \partial_{\alpha, \beta}$. Next, we define $\tilde{\partial} : \tilde{C}_k \rightarrow \tilde{C}_{k-1}$ by $\tilde{\partial} < \sigma > = < \partial \sigma >$. The boundary operator $\tilde{\partial} : \tilde{C}_*(\mathcal{R}^0) \rightarrow \tilde{C}_*(\mathcal{R}^0)$ is defined to be the direct sum of these boundary operators. (We abuse notations a bit here in order to avoid too many notations.)

Lemma 7.5. $\tilde{\partial}^2 = 0$. Hence $(\tilde{C}_*, \tilde{\partial})$ is a chain complex.

Proof. Consider $\sigma = (\Delta_j, f) \in C_j(S_\alpha)$. We have $\tilde{\partial}^2 \sigma = \sum_{\mu(\beta) < \mu(\alpha)} I_\beta$, where

$$I_\beta = < \partial_0 \partial_{\alpha, \beta} \sigma > + < \partial_{\alpha, \beta} \partial_0 \sigma > + \sum_{\mu(\alpha) > \mu(\gamma) > \mu(\beta)} < \partial_{\gamma, \beta} \partial_{\alpha, \gamma} \sigma >.$$

On the other hand, by (7.1) we have

$$\begin{aligned} (7.4) \quad \partial(\Delta_j \times_{S_\alpha} \hat{\mathcal{M}}_T(S_\alpha, S_\beta)) &= \partial \Delta_j \times_{S_\alpha} \hat{\mathcal{M}}_T(S_\alpha, S_\beta) + (-1)^{j + \dim S_\alpha} \Delta_j \times_{S_\alpha} \partial \hat{\mathcal{M}}_T(S_\alpha, S_\beta) \\ &= \partial \Delta_j \times_{S_\alpha} \hat{\mathcal{M}}_T(S_\alpha, S_\beta) + \\ &\quad (-1)^{j + \mu(\alpha)} \sum_{\mu(\alpha) > \mu(\gamma) > \mu(\beta)} \Delta_j \times_{S_\alpha} \hat{\mathcal{M}}_T(S_\alpha, S_\gamma) \times_{S_\gamma} \hat{\mathcal{M}}_T(S_\gamma, S_\beta). \end{aligned}$$

Multiplying this equation by $(-1)^{(j + \mu(\alpha) - \mu(\beta) - 1) + \mu(\beta)}$ and noting Remark 7.3 we then infer that $I_\beta = 0$. The desired identity follows. (This is similar to the situation in [9].) \square

Definition 7.6. We define the Bott-type Seiberg-Witten homology $FH_{b*}^{SW}(c)$ and cohomology $FH_b^{SW*}(c)$ to be the homology $H_*(\tilde{C}_*, \tilde{\partial})$ and cohomology $H^*(\tilde{C}_*, \tilde{\partial})$ of the complex $(\tilde{C}_*, \tilde{\partial})$. (Recall that c is the given $spin^c$ structure.)

Remark 7.7. In the above situation and in Section 9 of Part II, we can enlarge the space of singular chains $C_*(\mathcal{X})$ to allow all (Δ, f) , where Δ is a compact oriented smooth manifold with corners and f a continuous map from Δ to \mathcal{X} . Then we can avoid baricentric subdivisions and triangulations (we still divide out homotopy differences) and obtain equivalent homology and cohomology theories.

8. INVARIANCE I

Consider two chain complexes (C_*, ∂) and $(\bar{C}_*, \bar{\partial})$. A chain map of degree $m \in \mathbb{Z}$ between the two complexes consists of homomorphisms $F : C_k \rightarrow \bar{C}_{k+m}$ such that $F \cdot \partial = \bar{\partial} \cdot F$. A shifting homomorphism $F : H_*(C_*, \partial) \rightarrow H_*(\bar{C}_*, \bar{\partial})$ of degree $m \in \mathbb{Z}$ consists of homomorphisms $F : H_k(C_*, \partial) \rightarrow H_{k+m}(\bar{C}_*, \bar{\partial})$. One defines shifting homomorphisms for cohomologies in a similar way.

Our goal is to prove the following invariance result.

Main Theorem I. *The Bott-type Seiberg-Witten Floer homology and cohomology are diffeomorphism invariants modulo shifting isomorphisms.*

In other words, these homology and cohomology are independent of the metric h and generic parameter (π, λ) modulo shifting isomorphisms.

In this section, we construct the first kind of shifting homomorphisms which we need. In Part II, we present a second kind of shifting homomorphisms. We shall show that they provide inverses for each other.

Consider two metrics h_+ and h_- on Y and generic parameters (π_+, λ_+) for h_+ and (π_-, λ_-) for h_- respectively. We proceed to construct shifting homomorphisms between our homologies (cohomologies) constructed with respect to (h_+, π_+, λ_+) and (h_-, π_-, λ_-) respectively.

Choose a smooth path of metrics $h(t)$ on Y such that

$$h(t) = \begin{cases} h_-, & \text{if } t < -1, \\ h_+, & \text{if } t > 1, \end{cases}$$

a smooth path of $\pi(t) \in \Pi$

$$\pi(t) = \begin{cases} \pi_-, & \text{if } t < -1, \\ \pi_+, & \text{if } t > 1, \end{cases}$$

and a smooth function $\lambda(t) \in \mathbb{R}$ such that

$$\lambda = \begin{cases} \lambda_-, & \text{if } t < -1, \\ \lambda_+, & \text{if } t > 1. \end{cases}$$

The following lemma is an immediate consequence of Lemma C.1 in Appendix C.

Lemma 8.1. *Fix an $(A_0, \Phi_0) \in \mathcal{A}(X) \times \Gamma^+(X)$. For any $R > 0$, set $X_R = Y \times [-R, R]$ and*

$$\begin{aligned} \mathcal{S}_R = \{ & u = (A_0, \Phi_0) + (A, \Phi) : (A, \Phi) \in \Omega_1^1(X_R) \times \Gamma^+(X_R) \\ & \text{with } d^*A = 0 \text{ and } A|_{\partial X_R}(\frac{d}{dt}) = 0. \} \end{aligned}$$

Then \mathcal{S}_R is a global slice for the action of $\mathcal{G}_2^0(X_R)$ on $\mathcal{A}_1(X_R) \times \Gamma_1^+(X_R)$. In other words, $\mathcal{A}_1(X_R) \times \Gamma_1^+(X_R) = \mathcal{G}_2^0(X_R) \cdot \mathcal{S}_R$.

The following definition is a crucial construction.

Definition 8.2. Choose a nonzero $\Psi_0 \in \Gamma^-(X)$ with support contained in the interior of X_R . We define a smooth vector field Z on $\mathcal{A}_1(X_1) \times \Gamma_1^+(X_1)$ by

$$Z(g^*u) = g^{-1}\Psi_0,$$

for $g \in \mathcal{G}_2^0(X_R)$, $u \in \mathcal{S}_R$ and extend Z to $\mathcal{A}_{1,loc}(X) \times \Gamma_{1,loc}^+(X)$ by

$$Z(u) = Z(u|_{X_1}).$$

We endow X with the warped product metric determined by the family of metrics $h(t)$ and the standard metric on \mathbb{R} .

The following lemma is readily proved

Lemma 8.3. *Z is equivariant with respect to the action of $\mathcal{G}_{2,loc}^0$.*

Now we introduce the (perturbed) transition trajectory equation for $A = a + fdt, \Phi = \phi$.

$$(8.1) \quad \begin{aligned} \frac{\partial a}{\partial t} &= *F_a + d_Y f + \langle e_i \cdot \phi, \phi \rangle e^i + \nabla H_{\pi(t)}(a) + \epsilon b_0, \\ \frac{\partial \phi}{\partial t} &= -\not\partial_a \phi - \lambda(t)\phi + \epsilon Z, \end{aligned}$$

with additional parameters $\epsilon \in \mathbb{R}$ and b_0 , which is a smooth 1-form of compact support (it does not contain dt). The Hodge $*$ at time t in the equation is that of the metric $h(t)$. The perturbation term ϵZ is called a *spinor perturbation*. We have the following obvious, but crucial lemma.

Lemma 8.4. *The equation (8.1) is equivariant with respect to the action of $\mathcal{G}_{2,loc}^0$. Moreover, it has no reducible solution.*

A fundamental property of the Seiberg-Witten equation is a pointwise maximum principle for the spinor field, which is a consequence of the Weitzenböck formula (2.1). With the presence of Z , this principle no longer holds. Instead, we have the following result.

Lemma 8.5. *Let $(A, \Phi) = (a + fdt, \phi)$ be a solution of (8.1). Then there holds*

$$\|\Phi\|_{L^\infty} \leq CE(A, \Phi)$$

for a constant C depending only on the families $h(t), \pi(t), \lambda(t)$ and the geometry of Y .

Proof. Before proceeding with the proof, we first observe that by (2.11) and (8.1) the energy can be estimated in the following way

$$(8.2) \quad \begin{aligned} E(A, \Phi) &= 2 \lim_{t \rightarrow -\infty} \mathbf{cs}_{(\lambda, H)}(a(\cdot, t), \phi(\cdot, t)) \\ &\quad - 2 \lim_{t \rightarrow +\infty} \mathbf{cs}_{(\lambda, H)}(a(\cdot, t), \phi(\cdot, t)) + \int_X |Z|^2, \end{aligned}$$

$$(8.3) \quad \int_X |Z|^2 < C.$$

Using local Columb gauges provided by Lemma C.1 in Appendix C and a patching argument, we can perform a gauge transformation to convert (A, Φ) into a smooth solution. Since the L^∞ norm of Φ is invariant, we can assume that (A, Φ) is already smooth. Furthermore, we can assume that (A, Φ) is in temporal form.

For simplicity, we assume $\lambda(t) \equiv 0$ in the following argument. It is easy to modify it to handle $\lambda(t)$. Put

$$I_1 = \not\partial_a \phi, I_2 = *F_a + \langle e_i \cdot \phi, \phi \rangle e^i - \nabla H(a, \phi), I_3 = I_2 + \nabla H(a, \phi).$$

Then

$$\int (|I_1|^2 + |I_2|^2) = \frac{1}{2} E(A, \Phi, \Omega),$$

for any domain $\Omega \subset X$. For each $t \in \mathbb{R}$, we use the 3-dimensional Weitzenböck formula (2.1) on $Y \times \{t\}$ to derive

$$(8.4) \quad \partial_a I_1 = -\Delta_a \phi + \frac{\bar{s}}{4} \phi + \frac{1}{2} |\phi|^2 \phi + I_3 \phi,$$

where \bar{s} denotes the scalar curvature function of $(Y, h(t))$. Multiplying (8.4) by ϕ and integrating by parts, we infer

$$\int_{Y \times \{t\}} (|\nabla_a \phi|^2 + \frac{s}{4} |\phi|^2 + \frac{1}{2} |\phi|^4) \leq \int_{Y \times \{t\}} |I_1| \cdot |\partial_a \phi| + |I_3| |\phi|^2.$$

Using the Hölder inequality we then deduce that

$$\int_{Y \times \{t\}} (|\nabla_a \phi|^2 + \frac{1}{4} |\phi|^4) \leq C(1 + \int_{Y \times \{t\}} (|I_1|^2 + |I_2|^2)),$$

where C depends only on $\|\nabla H\|_{L^\infty}$, which can be estimated by appealing to Lemma 3.11. This last estimate implies

$$(8.5) \quad \int_{X_{R-2, R+2}} (|\nabla_a \phi|^2 + |\phi|^4) \leq C(1 + E(A, \Phi, X_{R-2, R+2}))$$

for any $R > 0$, where $X_{r, R} = Y \times [r, R]$.

Next we apply the Moser iteration to deduce the desired L^∞ estimate. Let $\xi : X \rightarrow [0, 1]$ be a cut-off function such that $\text{supp } \xi \subset X_{R-2, R+2}$ and $\xi(t) = 1$ for $t \in X_{R-1, R+1}$. By the 4-dimensional Weitzenböck formula (2.1) on X (recall that X is endowed with warped product metric), we have

$$(8.6) \quad D_A Z = -\Delta_A \Phi + \frac{s}{4} \Phi - \frac{1}{4} |\Phi|^2 \Phi.$$

Choosing $\xi^2 |\Phi|^p \Phi$ as a test function, where $p > 0$ will be determined later, we obtain

$$(8.7) \quad \begin{aligned} & \left(\int_X \xi^2 \nabla_A \Phi \cdot \nabla_A (|\Phi|^p \Phi) + \frac{s}{4} |\Phi|^{p+2} + \frac{1}{4} |\Phi|^{p+4} \right) \\ &= - \int_X (\xi^2 Z D_A (|\Phi|^p \Phi) + 2\xi \nabla \xi Z |\Phi|^p \Phi + 2\xi \nabla \xi \cdot \nabla \Phi |\Phi|^p \Phi). \end{aligned}$$

We have

$$\begin{aligned} \nabla_A (|\Phi|^p \Phi) &= |\Phi|^p \nabla_A \Phi + d|\Phi|^p \Phi \\ &= |\Phi|^p \nabla_A \Phi + p|\Phi|^{p-1} \frac{\langle \nabla_A \Phi, \Phi \rangle}{|\Phi|} \Phi, \nabla_A \Phi \cdot \nabla_A (|\Phi|^p \Phi) \\ &= |\Phi|^p |\nabla_A \Phi|^2 + p|\Phi|^{p-2} \langle \nabla_A \Phi, \Phi \rangle^2. \end{aligned}$$

On the other hand,

$$p\|\Phi\|^{p-2} \langle \nabla_A \Phi, \Phi \rangle^2 \leq C_\varepsilon \|\Phi\|^p \|\nabla_A \Phi\|^2 + \frac{Cp^2}{\varepsilon} \|\Phi\|^p$$

where $\varepsilon > 0$ is arbitrary. Similarly,

$$\begin{aligned} |D_A(|\Phi|^p \Phi)| &\leq |\Phi|^p |D_A \Phi| + |d|\Phi|^p \cdot \Phi| \\ &\leq C|\Phi|^p |\nabla_A \Phi| + p|\Phi|^p |\nabla_A \Phi| \\ &\leq C\varepsilon |\Phi|^p |\nabla_A \Phi|^2 + \frac{C}{\varepsilon} (p^2 + 1) |\Phi|^p, \\ |\nabla \xi| |\xi| |Z| |\Phi|^{p+1} &\leq C\xi^2 |\Phi|^p + C|\nabla \xi|^2 |\Phi|^{p+2}, \end{aligned}$$

and

$$|\nabla \xi| |\xi| |\nabla \Phi| |\Phi|^{p+1} \leq \varepsilon \xi^2 \|\Phi\|^p |\nabla \Phi|^2 + \frac{C}{\varepsilon} |\nabla \xi|^2 |\Phi|^{p+2}.$$

Choosing ε suitably, we deduce

$$\int \xi^2 |\Phi|^p |\nabla_A \Phi|^2 \leq C \int ((p^2 + 1) \xi^2 |\Phi|^p + |\nabla \xi|^2 |\Phi|^{p+2}).$$

Consequently,

$$\int_X \xi^2 |\Phi|^p |\nabla \Phi|^2 \leq C \int_X ((p^2 + 1) \xi^2 |\Phi|^p + |\nabla \xi|^2 |\Phi|^{p+2}),$$

or

$$\int_X \xi^2 |\nabla |\Phi|^{\frac{p+2}{2}}|^2 \leq C(p+1)^2 \int_X (\xi^2 |\Phi|^p + |\nabla \xi|^2 |\Phi|^{p+2}).$$

Now we set $w = |\Phi|^{(p+2)/2}$. By the Sobolev inequality and Hölder inequality, we arrive at

$$\begin{aligned} \|\xi w\|_{L^2}^2 &\leq C(p+1)^2 \int (|\xi \nabla w|^2 + |w \nabla \xi|^2) \\ &\leq C(p+1)^2 (\|\xi w\|_{L^2}^{\frac{p}{p+2}} + \|\nabla \xi\|_{L^2} \|w\|_{L^2}). \end{aligned}$$

Then we use the iteration process as presented in [11] to infer

$$\sup_{X_{R-1, R+1}} |\Phi| \leq C \|\Phi\|_{L^4(X_{R-2, R+2})}.$$

Combining it with (8.5) we are done. \square

We also have

Proposition 8.6. *An analogue of Proposition 6.2 for (8.1) holds.*

We have various configuration spaces and moduli spaces associated with (8.1) which are analogous to the spaces introduced in Section 4. All the analysis in Sections 4, 5 and 6 carries over. We shall be brief in formulating the relevant results.

Let e.g. \mathcal{R}_\pm denote the \mathcal{R} for $(h_\pm, \pi_\pm, \lambda_\pm)$. Consider $\alpha_- \in \mathcal{R}_-$, $\alpha_+ \in \mathcal{R}_+$ and $p_\pm \in \alpha_\pm$. We have the space of transition trajectories $\mathcal{N}(p_-, p_+)$ and the moduli spaces $\mathcal{M}^0(p_-, p_+)$, $\mathcal{M}_T(S_{\alpha_-}, S_{\alpha_+})$ etc.. We also have the various spaces of transition k -trajectories. A transition k -trajectory is a k -tuple (u_1, \dots, u_k) with a distinguished portion u_m , $1 \leq m \leq k$, such that $u_i \in \mathcal{N}(p_{i-1}, p_i)$ with $p_0 = p, p_k = p_+$; $p_i \in \mathcal{SW}^-, 0 \leq i \leq m-1$; $p_i \in \mathcal{SW}^+, m \leq i \leq k$. Now “proper” means that p_0, \dots, p_{m-1} belong to distinct gauge classes, and p_m, \dots, p_k belong to distinct gauge classes. The twisted time translation acts on all portions except the distinguished one. The other concepts regarding k -trajectories carry over easily to transition k -trajectories.

We have the following analogues of Proposition 4.12, Remark 4.14 and Proposition 6.12

Proposition 8.7. *Let ϵ be given. Then for generic b_0 (we shall say that (ϵ, b_0) is generic), transversality holds for the moduli spaces $\mathcal{M}^0(p_+, p_-)$ with $p_{\pm} \in \alpha_{\pm} \in \mathcal{R}_{\pm}$. Consequently, it is a smooth manifold of dimension $\text{ind } \mathcal{F}_{p_-, p_+} - \max \{\dim G_p, \dim G_q\} + 1$, where \mathcal{F}_{p_-, p_+} means the linearization of the operator in (8.1) (the transition Seiberg-Witten operator). Moreover, the spaces $\underline{\mathcal{M}}^0(p_0, \dots, p_k)$ with $p_0 \in SW_0^-, p_k \in SW_0^+$ are smooth manifolds.*

Proposition 8.8. *For generic (ϵ, b_0) , we have for all $\alpha_- \in \mathcal{R}_-, \alpha_+ \in \mathcal{R}_+$*

- (1) $\hat{\underline{M}}_T(S_{\alpha_-}, S_{\alpha_+})$ *has the structure of d -dimensional smooth oriented manifolds with corners, where $d = \text{ind } \mathcal{F}_{p_-, p_+} - \dim G_p - \dim G_q + 1$ for $p_- \in \alpha_-, p_+ \in \alpha_+$.*
- (2) *This structure is compatible with the natural stratification of $\hat{\underline{M}}_T(S_{\alpha_-}, S_{\alpha_+})$.*
- (3) *The temporal endpoint maps $\pi_- : \hat{\underline{M}}_T(S_{\alpha_-}, S_{\alpha_+}) \rightarrow S_{\alpha_-}$ and $\pi_+ : \hat{\underline{M}}_T(S_{\alpha_-}, S_{\alpha_+}) \rightarrow S_{\alpha_+}$ are smooth maps. But they are not fibrations in general.*

The proof of the first statement of Proposition 8.6 is analogous to the proof of Proposition 4.13 and Remark 4.15, because Lemma 8.4 rules out reducible transition trajectories. Note that instead of using holonomy perturbations we now use the perturbation b_0 as in [12]. (This perturbation is not time translation equivariant, and hence can't be applied in the construction of our homologies.)

Now we proceed to construct our first kind of shifting homomorphisms. Let O_{\pm} be the unique reducible elements in $\mathcal{R}_{h_{\pm}}$ respectively. We set

$$(8.8) \quad m_0 = \text{ind } \mathcal{F}_{O, O'} - 1.$$

The following lemma is analogous to Corollary 5.2.

Lemma 8.9. *We have*

$$(8.9) \quad \text{ind } \mathcal{F}_{p_-, p_+} = \mu_-([p_-]) - \mu_+([p_+]) + m_0 + \dim G_{p_+}.$$

Consequently,

$$(8.10) \quad \dim \mathcal{M}_T(S_{\alpha_-}, S_{\alpha_+}) = \mu_-(\alpha_-) - \mu_+(\alpha_+) + m_0 - \dim G_{\alpha_-} + 1.$$

Now we define homomorphisms $F : \tilde{C}_k(\mathcal{R}_-) \rightarrow \tilde{C}_{k+m_0}(\mathcal{R}_+)$. Consider $\langle \sigma \rangle \in C_j(S_{\alpha_-})$ with $\sigma = (\Delta_j, f)$. We choose a representative σ such that f and $f|_{\partial \Delta_j}$ are transversal to the endpoint maps π_- from the moduli spaces $\hat{\underline{M}}_T(\alpha_-, \alpha_+)$ and their boundaries for all $\alpha_+ \in \mathcal{R}_+$. For each α_+ with the corresponding moduli space nonempty, we follow the construction of the boundary operator $\partial_{\alpha, \beta}$ in Section 7 to obtain a singular chain $\sigma' \in C_{j'}(S_{\alpha_+})$ with $j' + \mu_+(\alpha_+) = j + \mu_-(\alpha_-) + m_0$. We define $F_{\alpha_-, \alpha_+}(\langle \sigma \rangle)$ to be $\langle \sigma' \rangle$. We define it to be zero if the moduli space is empty. Then we set

$$F(\langle \sigma \rangle) = \sum_{\mu_+(\alpha_+) \leq \mu_-(\alpha_-) + m_0} F_{\alpha_-, \alpha_+}(\sigma).$$

It is easy to see that we indeed have $F : \tilde{C}_k(\mathcal{R}_-) \rightarrow \tilde{C}_{k+m_0}(\mathcal{R}_+)$.

Proposition 8.10. *We have $\tilde{\partial} \cdot F = F \cdot \tilde{\partial}$, hence F is a chain map of degree m_0 from $(\tilde{C}_*(\mathcal{R}_-), \tilde{\partial})$ to $(\tilde{C}_*(\mathcal{R}_+), \tilde{\partial})$. The induced shifting homomorphisms between the homologies and cohomologies are denoted by F_* and F^* .*

Proof. The proof goes along the same lines as the proof of Lemma 7.3. To simplify notations, we argue here in terms of $\hat{\mathcal{M}}_T(S_i^-, S_k^+) = \cup_{\alpha_- \in S_i^-, \alpha_+ \in S_k^+} \hat{\mathcal{M}}_T(\alpha_-, \alpha_+)$. Analogous to (7.4) (the moduli spaces $\hat{\mathcal{M}}_T(\alpha_-, \alpha_+)$ are oriented in a way similar to Lemma 7.2) we have for $\sigma = (\Delta_j, f) \in C_j(S_i^-)$ and $k \leq i + m_0$

$$\begin{aligned} \partial(\Delta_j \times_{S_i^-} \hat{\mathcal{M}}_T(S_i^-, S_k^+)) &= \partial\Delta_j \times_{S_i^-} \hat{\mathcal{M}}_T(S_i^-, S_k^+) + \\ (-1)^{j+\dim S_i^-} \Delta_j \times_{S_i^-} \partial\hat{\mathcal{M}}_T(S_i^-, S_k^+) &= \partial\Delta_j \times_{S_i^-} \hat{\mathcal{M}}_T(S_i^-, S_k^+) + \\ (-1)^{j+i+m_0+1} \sum_{i>m \geq k-m_0} \Delta_j \times_{S_i^-} \hat{\mathcal{M}}_T(S_i^-, S_m^-) \times_{S_m^-} \hat{\mathcal{M}}_T(S_m^-, S_k^+) &+ \\ (-1)^{j+i+m_0+1} \sum_{i \geq m' - m_0 > k - m_0} \Delta_j \times_{S_i^-} \hat{\mathcal{M}}_T(S_i^-, S_{m'}^+) \times_{S_{m'}^+} \hat{\mathcal{M}}_T(S_{m'}^+, S_k^+). \end{aligned}$$

Clearly, this implies

$$\begin{aligned} \langle \partial_0 F_{S_i^-, S_k^+} \sigma \rangle &= - \langle F_{S_i^-, S_k^+} \partial_0 \sigma \rangle - \sum_{i>m \geq k-m_0} \langle F_{S_m^-, S_k^+} \partial_{S_i^-, S_m^-} \sigma \rangle - \\ &\quad \langle \partial_{S_{m'}^+, S_k^+} F_{S_i^-, S_{m'}^+} \sigma \rangle, \end{aligned}$$

where e.g. $\partial_{S_i^-, S_m^-}$ and $F_{S_m^-, S_k^+}$ are defined analogously to $\partial_{\alpha, \beta}$ and F_{α_-, α_+} . Summing over all k , we arrive at the desired chain homotopy property. \square

9. STABLE BOTT-TYPE AND STABLE EQUIVARIANT HOMOLOGY

Stable Bott-type Seiberg-Witten Floer homology

We consider

$$C_k^s = \oplus_{i+j=k} C_j(S_i \times S^1), C_s^k = \oplus_{i+j=k} C^j(S_i) \times S^1,$$

and the corresponding $\tilde{C}_k^s, \tilde{C}_s^k$, which are analogous to \tilde{C}_k, C^k .

Now the moduli spaces $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$ are replaced by $\hat{\mathcal{M}}_T(S_\alpha, S_\beta) \times S^1$, with endpoint maps:

$$\pi_+(u, s) = (\pi_+(u), s), \pi_-(u, s) = (\pi_-(u), s).$$

We have the boundary operator $\tilde{\partial}_s : \tilde{C}_k^s \rightarrow \tilde{C}_{k-1}^s$ analogous to $\tilde{\partial}$. (Note that the moduli spaces of trajectories and the submanifolds $S_\alpha \times S^1$ are both one dimension higher than in Section 7. The extra dimensions cancel each other in the construction of the boundary operator.) Similarly, we have $\tilde{\partial}_s^2 = 0$. Hence we can introduce the following stable Seiberg-Witten Floer homology and cohomology of Bott-type

Definition 9.1. We define $FH_{sb*}^{SW}(c) = H_*(\tilde{C}_*^s, \tilde{\partial}_s)$ and $FH_{sb*}^{SW*}(c) = H^*(\tilde{C}_*^s, \tilde{\partial}_s)$.

Stable equivariant Seiberg-Witten Floer homology

There is a diagonal action of S^1 on $\mathcal{R}^0 \times S^1$:

$$s^*(\alpha, s') = (s^*\alpha, s^{-1}s'),$$

where $s, s' \in S^1$, s is identified with the constant map from Y into S^1 with value s (a constant gauge), and $s^*[p]_0 = [s^*p]_0$. This action induces an action of S^1 on singular chains in $\mathcal{R}^0 \times S^1$: if $\sigma = (\Delta, f)$ is a j -chain, and $s \in S^1$, then $(s^*\sigma)(z) = (\Delta_j, s^*f(\cdot))$. Passing to quotients, we obtain an action of S^1 on $\tilde{C}_*(\mathcal{R}^0 \times S^1)$. A j -cochain class $\omega \in \tilde{C}^*(\mathcal{R}^0 \times S^1)$ is called *equivariant* (or *invariant*), provided that $\omega(s^* \langle \sigma \rangle) = \omega(\langle \sigma \rangle)$ for all $\langle \sigma \rangle \in \tilde{C}_*(\mathcal{R}^0)$ and $s \in S^1$.

We define $\tilde{C}_e^j(S_i \times S^1)$ to be the free abelian group of equivariant j -cochain classes on $S_i \times S^1$ and set

$$\tilde{C}_e^k = \oplus_{i+j=k} C_e^j(S_i \times S^1).$$

Now let $\tilde{\partial}_e^*$ be the restriction of $\tilde{\partial}_s^*$ to equivariant cochain classes.

Lemma 9.2. *The endpoint maps π_\pm defined on the moduli spaces $\hat{\mathcal{M}}_T(S_\alpha, S_\beta) \times S^1$ are S^1 -equivariant, where the S^1 action on $\hat{\mathcal{M}}_T(S_\alpha, S_\beta) \times S^1$ is induced from the following action: $s \in S^1$ acts on (u, s') to yield $(s^*u, s^{-1}s')$. (If we work with the model $\hat{\mathcal{M}}^0(p, q)$, then the S^1 action is induced from that on $\hat{\mathcal{M}}_T(S_\alpha, S_\beta)$ through the temporal transformation.)*

We omit the easy proof. As a consequence of this lemma, the operator $\tilde{\partial}_e^*$ has equivariant cochain classes as values. Hence we have $\tilde{\partial}^* : \tilde{C}_e^k \rightarrow \tilde{C}_e^{k+1}$.

Definition 9.3. The stable equivariant Seiberg-Witten Floer cohomology $FH_{se}^{SW*}(c)$ is defined to be the homology $H_*(\tilde{C}_e^*, \tilde{\partial}_e^*)$ of the complex $(\tilde{C}_e^*, \tilde{\partial}_e^*)$. The stable equivariant Seiberg-Witten Floer homology $FH_{se*}^{SW}(c)$ is defined to be the cohomology $H^*(\tilde{C}_e^*, \tilde{\partial}_e^*)$ of the complex $(\tilde{C}_e^*, \tilde{\partial}_e^*)$.

INVARIANCE II

The purpose of this section is to prove the following two results.

Main Theorem II. *The stable Bott-type Seiberg-Witten Floer homology and cohomology are diffeomorphism invariants up to shifting isomorphisms.*

Main Theorem III. *The stable equivariant Seiberg-Witten Floer homology and cohomology are diffeomorphism invariants up to shifting isomorphisms.*

We start with

Definition 10.1. Let $\mathcal{G}_{2,loc}$ act on $(\mathcal{A}_{1,loc} \times \Gamma_{1,loc}^+) \times S^1$ in the following fashion:

$$(gg_0)^*(u, s) = ((gg_0)^*u, g_0^{-1}s),$$

where $g \in S^1$ (the group of constant gauges), $g_0 \in \mathcal{G}_{2,loc}^0$, and $s \in S^1$.

Definition 10.2. We define a smooth vector field Z_e on $(\mathcal{A}_{1,loc} \times \Gamma_{1,loc}^+) \times S^1$ as follows

$$Z_e(u, s) = sZ(u),$$

where Z is the vector field given by Definition 8.3.

The following lemma is readily proved.

Lemma 10.3. Z_e is equivariant with respect to the action of $\mathcal{G}_{2,loc}$.

Now we introduce the following stable version of the transition trajectory equation (8.1) for $A = a + fdt$, $\Phi = \phi$ and $s \in S^1$. Its solutions will be called *stable transition trajectories*.

$$(10.1) \quad \begin{aligned} \frac{\partial a}{\partial t} &= *F_a + d_Y f + \langle e_i \cdot \phi, \phi \rangle e^i + \nabla H_{\pi(t)}(a) + \epsilon b_0, \\ \frac{\partial \phi}{\partial t} &= -\partial_a \phi - \lambda(t)\phi + \epsilon s Z_e(A, \Phi), \end{aligned}$$

where $s \in S^1$ and ϵ, b_0 are the same as in (8.1).

Now consider $(h_{\pm}, \pi_{\pm}, \lambda_{\pm})$ as in Section 8. For $\alpha_{\pm} \in \mathcal{R}_{\pm}$ and $p_{\pm} \in \alpha_{\pm}$, we consider $\alpha_{\pm} \times S^1$ and set

$$\mathcal{N}(p_- \times S^1, p_+ \times S^1) = \{(u, s) : (u, s) \text{ is a stable transition trajectory converging to } p_{\pm} \text{ at } \pm \infty.\}$$

We have moduli spaces $\mathcal{M}^0(p_- \times S^1, p_+ \times S^1)$, $\mathcal{M}_T(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1)$ etc.. We also have the various spaces of stable transition k -trajectories. A stable transition k -trajectory is a pair $((u_1, \dots, u_k), s)$ with $s \in S^1$ and a distinguished portion u_m , $1 \leq m \leq k$, such that $(u_i, s) \in \mathcal{N}(p_{i-1} \times S^1, p_i \times S^1)$ with $p_0 = p_-, p_k = p_+$; $p_i \in \mathcal{SW}^-, 0 \leq i \leq m-1$; $p_i \in \mathcal{SW}^+, m \leq i \leq k$. Now “proper” means that p_0, \dots, p_{m-1} belong to distinct gauge classes, and p_m, \dots, p_k belong to distinct gauge classes. The twisted time translation is easily defined and acts on all portions except the distinguished one. The concept of consistent stable transition k -trajectories is also easily defined. Finally, note that $\pi_{\pm}(u, s)$ is defined to be $(\pi_{\pm}(u), s)$.

We have the following analogues of Proposition 4.12, remark 4.14 and Proposition 6.13.

Proposition 10.4. Let ϵ be given. Then for generic b_0 (we shall say that (ϵ, b_0) is generic), transversality holds for the moduli spaces $\mathcal{M}^0(p_+ \times S^1, p_- \times S^1)$ for $p_{\pm} \in \alpha_{\pm} \in \mathcal{R}_{\pm}$. Consequently, it is a smooth manifold of dimension $\mu_-(\alpha_-) - \mu_+(\alpha_+) + m - \dim G_{\alpha_-} + 1$. Moreover, the spaces $\underline{\mathcal{M}}^0(p_0 \times S^1, \dots, p_k \times S^1)$ with $p_0 \in \alpha_-, p_k \in \alpha_+$ are smooth manifolds.

Proposition 10.5. For generic (ϵ, b_0) , we have for all $\alpha_- \in \mathcal{R}_{h_-}, \alpha_+ \in \mathcal{R}_{h_+}$

(1) $\hat{\mathcal{M}}_T(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1)$ has the structure of d -dimensional smooth oriented manifolds with corners, where $d = \mu_-(\alpha_-) - \mu_+(\alpha_+) + m - \dim G_{\alpha_-} + 1$.

(2) This structure is compatible with the natural stratification of $\hat{\mathcal{M}}_T(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1)$.

(3) The temporal endpoint maps $\pi_{\pm} : \hat{\mathcal{M}}_T(S_{\alpha_-} \times S^1, S_{\alpha_+} \times S^1) \rightarrow S_{\alpha_{\pm}} \times S^1$ are S^1 -equivariant smooth maps, where the action of S^1 on the moduli space is induced

by the following action: $s \in S^1$ acts on (u, s') to yield $(s^*u, s^{-1}s')$. But they are not fibrations in general.

To prove Main Theorem II, we construct chain maps $F^+ : \tilde{C}_*^{s+} \rightarrow C_*^{s-}$ as in Section 8. We need to show that they induce isomorphisms between the homologies and cohomologies. The arguments consist of three steps.

Step 1

Consider the equation (10.1) with parameters determined by the construction of F^- and the same equation with parameters determined by the construction of F^+ . We interpolate (glue) the former with the latter to obtain a new equation of a similar type, which we call the *glued transition equation*. There is an interpolation parameter ρ in this equation. When $\rho \rightarrow 0$, the equation breaks into the original two equations. Fix some $\rho_0 > 0$. Using the glued transition equation with this parameter we construct a chain map $F_+^- : \tilde{C}_*^- \rightarrow \tilde{C}_*^-$ of degree zero in the same way as e.g. the construction of F^- . (The compactified moduli spaces for this equation has the same structure as described in Proposition 10.7.) Employing the enlarged glued transition equation in which the parameter ρ varies in the range $0 < \rho \leq \rho_0$, we then obtain a chain map $\Theta : \tilde{C}_*^- \rightarrow \tilde{C}_*^-$ of degree one (instead of zero, because the corresponding moduli spaces are one dimension higher than before). (The compactified moduli spaces have a similar structure to that described in Proposition 10.7.) Arguing as in the proofs of Lemma 7.2 and Proposition 8.10, we infer that this map is a chain homotopy between $F^+ \cdot F^-$ and F_+^- , i.e. $F^+ \cdot F^- - F_+^- = \tilde{\partial} \cdot \Theta + \Theta \cdot \tilde{\partial}$.

Step 2

Next we make use of the parameter ϵ in (8.1), which enters the glued transition equation. (There are actually two such parameters in the glued transition equation. We can turn them into one parameter.) When $\epsilon = 0$, the glued transition equation reduces to the trajectory equation (4.1) with some holonomy and λ perturbations. We can incorporate additional holonomy perturbations to ensure transversality between irreducible Seiberg-Witten points or one irreducible and one reducible. The transversality between two reducibles follows from Lemma C.2. Then we can construct a chain map $F_0^- : \tilde{C}_*^- \rightarrow \tilde{C}_*^-$ of degree zero. It is easy to see that it induces the identity isomorphism of the homology and cohomology.

Allowing ϵ to vary from the given generic value to zero, we obtain another enlarged glued transition equation. Using it we then obtain a chain homotopy between F_+^- and F_0^- . It follows that $F_*^+ \cdot F_*^- = Id, F_*^{-*} \cdot F_*^{+*} = Id$.

Step 3

A similar construction with the roles of F^- and F^+ reversed yields $F_*^- \cdot F_*^+ = Id, F_*^{+*} \cdot F_*^{-*} = Id$.

We have proved Main Theorem II. The same arguments apply to Main Theorem III because of the equivariance of the constructions.

Let Y be a rational homology sphere with a given metric h and c a $spin^c$ structure on Y as before. Set $C_i = \mathbb{Z}\{\alpha \in \mathcal{R}^* | \mu(\alpha) = i\}$. We define a boundary operator $\partial : C_i \rightarrow C_{i-1}$ in terms of the moduli spaces $\widetilde{\mathcal{M}}(\alpha, \beta)$, or equivalently $\widetilde{\mathcal{M}}(p, q)$, $p \in \alpha, q \in \beta$, where the tilde means quotient by the time translation action. For a generic pair (π, λ) , $\widetilde{\mathcal{M}}(\alpha, \beta)$ is a compact oriented manifold of zero dimension, provided that $\mu(\alpha) - \mu(\beta) = 1$. The orientation (sign) is given by pairing the orientation of $\mathcal{M}(\alpha, \beta)$ provided by Proposition 5.3 with its orientation induced by the time translation action. For $\alpha \in C_i$ we then set

$$(A.1) \quad \partial\alpha = \sum_{\mu(\alpha) - \mu(\beta) = 1} \# \widetilde{\mathcal{M}}(\alpha, \beta) \beta,$$

where $\# \widetilde{\mathcal{M}}(p, q)$ is the algebraic sum of $\widetilde{\mathcal{M}}(\alpha, \beta)$.

The compactification of the moduli spaces $\widetilde{\mathcal{M}}(\alpha, \gamma)$ with $\mu(\alpha) - \mu(\gamma) = 2$ is similar to the results in Section 6. For dimensional reasons, no trajectory connecting to the reducible point appears in the compactification. Using these compactified moduli spaces and the consistency of orientation (Proposition 5.3) we obtain

Lemma A.1. $\partial^2 = 0$.

Definition A.2. The Seiberg-Witten Floer homology $FH_*^{SW}(c, h, \pi, \lambda)$ and cohomology $FH^{SW*}(c, h, \pi, \lambda)$ for the $spin^c$ structure c and the parameters h, π, λ are defined to be the homology and cohomology of the chain complex (C_*, ∂) .

APPENDIX B EQUIVARIANT SEIBERG-WITTEN FLOER HOMOLOGY

There are two possible versions. The first uses equivariant singular cochains. The construction is similar to equivariant construction in Section 9 of Part II. We use equivariant singular cochains on S_α instead of $S_\alpha \times S^1$. The resulting homology and cohomology will be denoted by FH_{e*}^{SW} and FH_e^{SW*} .

The second version uses equivariant differential forms. Let G be a compact Lie group, \mathfrak{g} its Lie algebra and M a G -manifold. Let $\mathbb{C}[\mathfrak{g}^*]$ denote the algebra of complex valued polynomial function on \mathfrak{g} . We define the space of equivariant differential forms

$$\Omega_G(X) = (\Omega^*(X) \otimes \mathbb{C}[\mathfrak{g}])^G$$

to be the subalgebra of G -invariant elements in the algebra $\Omega^*(M) \otimes \mathbb{C}[\mathfrak{g}]$. The algebra $\Omega^*(M) \otimes \mathbb{C}[\mathfrak{g}]$ has a \mathbb{Z} -grading defined by

$$\deg(w \otimes z) = 2 \deg(z) + \deg(w),$$

where $w \in \Omega(M)$ and $z \in \mathbb{C}[\mathfrak{g}]$. We define a differential d_G :

$$(d_G \alpha)(X) = d(\alpha(X)) - \iota(X)(\alpha(X)),$$

where $\alpha \in \Omega_G(M)$ and $\iota(X)$ denotes the contraction by the vector field X_M induced by an element $X \in \mathfrak{g}$. We have $d_G^2 = 0$.

To define the desired equivariant homology and cohomology for given generic parameters (π, λ) , we set

where j denotes the degree of differential forms. (The S_i were defined in Section 7.) We define operators $\partial_r : C^{i,j} \rightarrow C^{i+r,j-r+1}$ by

$$\partial_r \omega = \begin{cases} d_{S^1} \omega, & \text{if } r = 0, \\ (-1)^j \pi_{+*} \pi_-^*(\omega) & \text{otherwise,} \end{cases}$$

where the map π_{+*} is integration along the fiber of the fibration $\pi_+ : \hat{\mathcal{M}}_T(S_i, S_{i+r}) \rightarrow S_{i+r}$ (cf. [3]), and π_- refers to the fibration $\pi_- : \hat{\mathcal{M}}_T(S_i, S_{i+r}) \rightarrow S_{i+r}$. ($\hat{\mathcal{M}}_T(S_i, S_j) \equiv \cup_{\alpha \in S_i, \beta \in S_j} \hat{\mathcal{M}}_Y(\alpha, \beta)$.) We set

$$\partial_{S^1} = \sum \partial_r.$$

Using the compactification results in Section 6 and the arguments in [4], one easily deduces

Lemma B.1. $\partial_{S^1}^2 = 0$.

Definition B.2. The deRham type equivariant Seiberg-Witten Floer homology $FH_{de*}^{SW}(c, h, \pi, \lambda)$ for a given metric h and generic parameters h, π, λ is defined to be the homology of the chain complex (C^*, ∂_{S^1}) . The corresponding cohomology FH_{de}^{SW*} is defined by a dual construction, cf. [3].

APPENDIX C TWO ANALYSIS LEMMAS

First, we prove a result on local Columb gauge fixing. We assume that Y is a rational homology sphere. Set $X_{r,R} = Y \times [r, R]$ for $r < R$ and fix a point $x_0 \in X_{r,R}$.

Lemma C.1. *For any $A \in \mathcal{A}_l(X_{r,R})$ with $l \geq 1$, there exists a unique gauge $g \in \mathcal{G}_{l+1}(X_{r,R})$ with $g(x_0) = 1$ such that $\tilde{A} = g^*(A)$ satisfies*

$$(C.1) \quad \begin{aligned} d^* \tilde{A} &= 0, \\ A(\nu) &= 0 \text{ on } \partial X_{r,R}, \end{aligned}$$

where ν denotes the unit outer normal of $\partial X_{r,R}$. Moreover, we have

$$\|A\|_{1,2} \leq C \|F_A\|_{0,2}$$

for a positive constant C depending only on Y, r and R .

Proof. The associated gauge fixing equation is

$$(C.2) \quad \begin{aligned} d^*(g^{-1}dg) + d^*A &= 0, \\ g^{-1}dg(\nu) + A(\nu) &= 0 \text{ on } \partial X_{r,R}. \end{aligned}$$

If we choose $g = e^f$, then the equation reduces to

$$(C.3) \quad \begin{aligned} d^*df + d^*A &= 0, \\ \frac{\partial f}{\partial \nu} + A(\nu) &= 0 \text{ on } \partial X_{r,R}. \end{aligned}$$

It is clear that a solution $f \in C^0(Y, \mathbb{R})$ with $f(x_0) = 0$ exists

Now assume that there are g and g_1 satisfying (C.2) with $g(x_0) = g_1(x_0) = 1$. Without loss of generality, we may assume that $g_1 = id$. Taking two copies of $X_{r,R}$ and gluing them along their common boundary, we obtain the Riemannian manifold $Y \times S^1$. The two copies of g then yield a solution $g_0 : Y \times S^1 \rightarrow S^1$ of the harmonic equation $d^*(g^{-1}dg) = 0$. Clearly, for each $y \in Y$, $g_0(y, \cdot)$ is a map from S^1 to S^1 with degree zero, *i.e.* $\int_{\{y\} \times S^1} g_0^{-1}dg_0 = 0$. On the other hand, $g_0^{-1}dg_0$ defines an element ω in H_{de}^1 . Since Y is a rational homology sphere, we deduce $\omega = 0$ and that g_0 is constant. Since $g(x_0) = 0$, we infer $g \equiv 1$.

Next consider an A satisfying (i) and (ii) in Lemma C.1. As in the above argument, A leads to a one form A_0 on $Y \times S^1$ satisfying $d^*A_0 = 0$. Since Y is a rational homology sphere, it is easy to see from the construction of A_0 that the harmonic part of A_0 in its Hodge decomposition is zero. Hence we have $A_0 = d^*\omega$ for some two form $\omega \in \Omega_2^2(Y \times S^1)$. Consequently, $dd^*\omega = F_{A_0}$. We deduce

$$\|A_0\|_{1,2} = \|d^*\omega\|_{1,2} \leq C\|F_{A_0}\|_{0,2}.$$

This implies the desired estimate. \square

Next we prove transversality at reducible trajectories.

Lemma C.2. *Let (π, λ) be a pair of good parameters such that $\nabla^2 H$ is sufficiently small. Choose δ_-, d_+ small enough (but positive) in the set-up of Definition 4.4. Then the operator $\mathcal{F}_{p,q}$ for $p, q \in \mathcal{O}$ at a Seiberg-Witten trajectory is onto.*

Proof. Consider $\mathcal{F} = \mathcal{F}_{p,q}$ at a trajectory (A_0, Φ_0) . By the proof of Lemma 4.12, we can assume $p = q = (a_0, 0)$ and $(A_0, \Phi_0) \equiv p$. The formal adjoint \mathcal{F}^* of $\mathcal{F} = \mathcal{F}_{p,q}$ with respect to the product (4.4) is given by

$$(C.4) \quad \mathcal{F}^*(v) = -\frac{d}{dt}v - \begin{pmatrix} *db + df - \nabla^2 H(a_0) \cdot b \\ -\tilde{\partial}_{a_0}\psi - \lambda\psi \\ d^*b - \delta'_F f \end{pmatrix} - \delta'(t)v$$

for $v = (\psi, b, f)$. The surjectivity of \mathcal{F} is equivalent to the vanishing of the kernel of \mathcal{F}^* . Let v satisfy $\mathcal{F}^*v = 0$. Then we have

$$(C.5) \quad \frac{d}{dt} \begin{pmatrix} b \\ f \end{pmatrix} + \begin{pmatrix} *db + df \\ d^*b \end{pmatrix} + \begin{pmatrix} \delta'_F b + \nabla^2 H(a_0) \cdot b \\ 0 \end{pmatrix} = 0.$$

We define the operator L by

$$L \begin{pmatrix} b \\ f \end{pmatrix} = \begin{pmatrix} *db + df \\ d^*b \end{pmatrix}.$$

L is formally self-adjoint and satisfies $L^2 = \Delta$, where Δ denotes the Hodge Laplacian. Let $\{\xi_i = \begin{pmatrix} b_i \\ f_i \end{pmatrix}\}$ be a complete L^2 orthonormal system of eigenvectors of L with $L\xi_i = \lambda_i \xi_i$. From the above discussion we deduce

$$\Delta b_i = \lambda_i^2 b_i, \Delta f_i = \lambda_i^2 f_i.$$

Now we write

$$\begin{pmatrix} b \\ f \end{pmatrix} = \sum_{i=1}^{+\infty} l_i(t) \xi_i.$$

Then it follows from C.5 that

$$(C.6) \quad \sum l'_i(t)\xi_i + \lambda_i l_i(t)\xi_i + \begin{pmatrix} \delta'_F b + \nabla^2 H(a_0) \cdot b \\ 0 \end{pmatrix} = 0.$$

Assume $\lambda_j = 0$ for a j . Then there holds $b_j = 0$, for Y is a rational homology sphere and hence supports no nontrivial harmonic 1-form. Consequently, f_j is a nonzero constant. Then we deduce $l'_j(t) \equiv 0$, hence l_j is a constant. But (b, f) is L^2 integrable, which forces l_j to be zero. We conclude that the above expansion of (b, f) does not contain terms with zero eigenvalue. Using the elementary arguments in e.g. [16] it is then easy to show that (b, f) must vanish, provided that $\nabla^2 H$, δ_+ and δ_- have been chosen small enough. Using the same arguments one also infers that ψ vanishes. Thus $v = 0$. \square

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